Today

4 Series

1) nth Triangular #
2) nth Harmonic #
3) Inverse Squares
4) Geometric

Big 3 Inequalities

1) Cauchy-Schwarz
2) AM-GM
3) Jensen's Inequality (+ Convexity)

Announce

- Rev. hw due
- Hw 1 out
- Scribe notes updates
- Overrides
- tidy notes

4 Series to Know

① nth Triangular Number : $\sum_{i=1}^{n} i = \Theta(n^2)$

$$1+2+3+\dots+n-2+n-1+n = \frac{n(n+1)}{2}$$

$\underbrace{n+1 \quad n+1 \quad n+1 \quad n+1}_{\times\frac{n}{2}\text{ times}}$

Really for even n

For odd n get middle term

$$\left(\frac{n-1}{2}\right)(n+1) + \frac{n+1}{2}$$

$$= \frac{n(n+1)}{2}$$

② nth Harmonic Number : $H_n = \sum_{i=1}^{n} \frac{1}{i} = \Theta(\log n)$

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\dots$$

$$\le \underbrace{\frac{1}{1}}_{=1}+\underbrace{\frac{1}{2}+\frac{1}{2}}_{=1}+\underbrace{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}_{=1}+\underbrace{\frac{1}{8}+\dots}_{=1}+\frac{1}{n}$$

$$\ge \underbrace{\frac{1}{2}}_{=\frac{1}{2}}+\underbrace{\frac{1}{4}+\frac{1}{4}}_{=\frac{1}{2}}+\underbrace{\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}}_{=\frac{1}{2}}+\dots$$

$$\Theta(\log n)$$

③ Inverse Squares : $\sum_{i=1}^{n} \frac{1}{i^2} = \Theta(1)$

Add/Subtract: $$\frac{1}{i(i-1)} = \frac{1-i+i}{i(i-1)} = \frac{i-(i-1)}{i(i-1)} = \frac{1}{i-1} - \frac{1}{i}$$

Telescoping: $$\sum_{i=2}^{n} \frac{1}{i-1} - \frac{1}{i} = \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \dots \cancel{\frac{1}{n-1}} - \frac{1}{n}$$

$$= 1 - \frac{1}{n}$$

$$\sum_{i=1}^{n} \frac{1}{i^2} \le 1 + \sum_{i=2}^{n} \frac{1}{i(i-1)} = 1 + \sum_{i=2}^{n} \frac{1}{i-1} - \frac{1}{i} = 1 + 1 - \frac{1}{n} \le 2$$

$\Omega(1)$ trivial

④ Geometric : $\sum_{i=0}^{\infty} r^i = \Theta(1)$ (for $r \in (0,1)$)

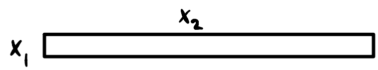
Series as Variable : Let $S := \sum_{i=1}^{\infty} r^i$ so $S = 1+r+r^2+\dots = 1+r\cdot S$

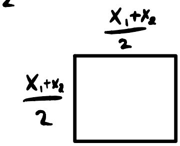
so $S(1-r) = 1 \rightarrow S = \frac{1}{1-r}$

2 Geometrically "Obvious" Facts and The "Big 3" Inequalities

Puzzle ①: "Square maximizes area"

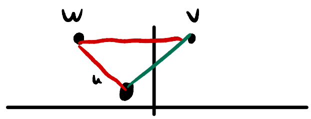
given a rectangle w/ side lengths x_1, x_2
then square w/ side lengths $\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}$
has larger area





Puzzle ②: "triangle inequality"

$\forall$ 3 points $u,v,w \in \mathbb{R}^2$, $u \to v \leq u \to w \to v$



Each puzzle really an algebra fact

① $\prod_i x_i \leq \left(\frac{\sum_1^2 x_i}{2}\right)^2 \iff \left(\prod_i x_i\right)^{1/2} \leq \frac{\sum_1^2 x_i}{2}$

② Recall $d(u,v) := \sqrt{\sum_i^2 (u_i - v_i)^2}$

Translation invariant: $d(u+x, v+x) = d(u,v) \quad \forall x \in \mathbb{R}^2$

$$d(u,v) \leq d(u,w) + d(w,v) \quad \forall u,v,w$$

$$\Updownarrow \text{(TI)}$$

$$d(0, v-u) \leq d(0, w-u) + d(0, v-w) \quad \forall u,v,w$$

$$\Updownarrow \quad \text{Let } \left.\begin{array}{l} \tilde{u} = w-u \\ \tilde{v} = v-w \end{array}\right] \to \tilde{u}+\tilde{v} = v-u$$

$$d(0, \tilde{u}+\tilde{v}) \leq d(0,\tilde{u}) + d(0,\tilde{v}) \quad \forall \tilde{u},\tilde{v}$$

$$\Updownarrow$$

$$d(0, u+v) \leq d(0,u) + d(0,v) \quad \forall u,v$$

$$\Updownarrow$$

$$\sqrt{\sum_i^2 (u_i+v_i)^2} \leq \sqrt{\sum_i^2 u_i^2} + \sqrt{\sum_i^2 v_i^2} \quad \forall u,v \overset{\text{rearrange}}{\iff} \sum_i^2 u_i v_i \leq \sqrt{\sum_i^2 u_i^2 \cdot \sum_i^2 v_i^2} \quad \forall u,v$$

$$\iff \langle u,v\rangle \leq \sqrt{\langle u,u\rangle\langle v,v\rangle} \quad \forall u,v$$

$\sum_i^2 (u_i+v_i)^2 \leq \sum_i^2 u_i^2 + \sum_i^2 v_i^2 + 2\sqrt{\sum_i^2 u_i^2 \sum_i^2 v_i^2}$

$\Updownarrow$

$\sum_i^2 2u_i v_i \leq 2\sqrt{\sum_i^2 u_i^2}\sqrt{\sum_i^2 v_i^2}$

Both provable w/ non-negativity of squares

↳ $x^2 \ge 0 \ \forall x \in \mathbb{R}$

(1) $\sqrt{x_1 \cdot x_2} \le \frac{x_1 + x_2}{2}$

$\Updownarrow$

$4 x_1 x_2 \le x_1^2 + 2 x_1 x_2 + x_2^2$

$\Updownarrow$

$0 \le x_1^2 - 2 x_1 x_2 + x_2^2$

$\Updownarrow$

$0 \le (x_1 - x_2)^2$

(2) $u_1 v_1 + u_2 v_2 \le \sqrt{(u_1^2 + u_2^2)(v_1^2 + v_2^2)}$

$\Updownarrow$

$\cancel{u_1^2 v_1^2} + 2 u_1 v_1 u_2 v_2 + \cancel{u_2^2 v_2^2} \le \cancel{u_1^2 v_1^2} + u_1^2 v_2^2 + u_2^2 v_1^2 + \cancel{u_2^2 v_2^2}$

$\Updownarrow$

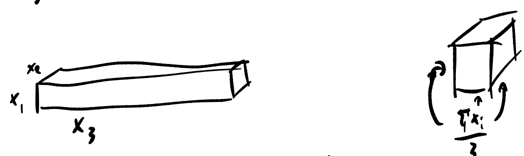
$0 \le u_1^2 v_2^2 + u_2^2 v_1^2 - 2 u_1 v_1 u_2 v_2$

$\Updownarrow$

$0 \le (u_1 v_2 - u_2 v_1)^2$

Algorithms often about higher-dimensional geometry

Puzzle (1'): "hypercube maximizes area"



In Algebra: $\underbrace{\left(\prod_{i=1}^{k} x_i\right)^{1/k}}_{GM} \le \underbrace{\frac{\sum_{i=1}^{k} x_i}{k}}_{AM} \quad \forall x_i \in \mathbb{R}_{\ge 0}$

$\boxed{\text{AM-GM}}$

Puzzle (2'): "triangle inequality (in higher dimensions)"

In Algebra: $\sum_{i=1}^{k} u_i v_i \le \sqrt{\left(\sum_{i=1}^{k} u_i\right)^2 \left(\sum_{i=1}^{k} v_i\right)^2} \quad \forall u, v \in \mathbb{R}^k$

$\boxed{\text{Cauchy-Schwarz}}$

Proving Cauchy-Schwarz

By induction on k

Base case of $n=2$ already done

Inductive step

$$\underbrace{u_1 v_1 + u_2 v_2 + \dots + u_{k-1} v_{k-1}}_{IH} + u_k v_k \le \underbrace{\sqrt{\sum_{i=1}^{k-1} u_i^2}}_{u_1'} \underbrace{\sqrt{\sum_{i=1}^{k-1} v_i^2}}_{v_1'} + \underbrace{u_k}_{u_2'} \underbrace{v_k}_{v_2'}$$

$$\le \sqrt{(u_1')^2 + (u_2')^2} \sqrt{(v_1')^2 + (v_2')^2} \quad (IH)$$

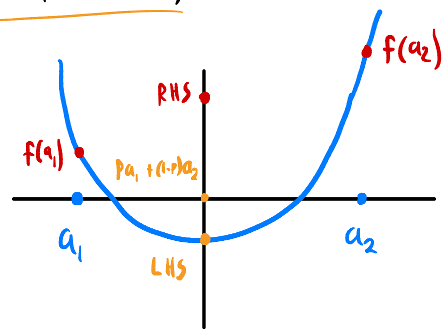
$$= \sqrt{\sum_1^k u_i^2} \sqrt{\sum_1^k v_i^2}$$

Proving AM-GM

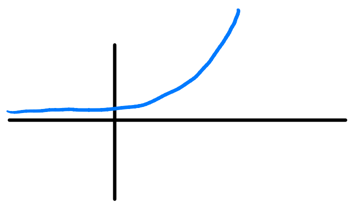
$f: \mathbb{R} \to \mathbb{R}$ is <u>convex</u> if $\forall\ p \in [0,1]$ and $a_1, a_2 \in \mathbb{R}$

$$f(pa_1 + (1-p)a_2) \leq p f(a_1) + (1-p) f(a_2)$$

"Bowl shape"



Eg. e^x



Jensen's Inequality

Let $p_1, \dots, p_k \in [0,1]$, s.t. $\sum_i p_i = 1$ and $a_1, \dots, a_k \in \mathbb{R}$, f convex

Then $f\left(\sum_i p_i a_i\right) \leq \sum_i p_i f(a_i)$ $\forall$ convex f

Proof of AM-GM w/ Jensen's

Let $a_i = \ln(x_i)$, $f = e^x$ and $p_i = \frac{1}{k}$ $\forall i$

Jensen's gives

$$e^{(\ln(x_1) + \ln(x_2) + \dots)/k} \leq \sum_i \frac{1}{k} e^{\ln(x_i)}$$

So $\left(\prod x_i\right)^{1/k} \leq \sum x_i / k$

Proving Jensen's Inequality

By induction on k

Base Case: $k=2 \to$ defn. of convexity

Inductive Step:

Let $p = \sum_{i=1}^{k-1} p_i$ so $p_k = 1-p$

$$f(p_1 a_1 + \ldots + p_k a_k) = f\left(p\left(\frac{p_1 a_1}{p} + \ldots + \frac{p_{k-1} a_{k-1}}{p}\right) + (1-p) a_k\right)$$

$$\leq p\, f\left(\frac{p_1 a_1}{p} + \ldots + \frac{p_{k-1} a_{k-1}}{p}\right) + (1-p) f(a_k) \quad (IH)$$

$$\leq p\left(\frac{p_1}{p} f(a_1) + \ldots + \frac{p_{k-1}}{p} f(a_{k-1})\right) + p_k a_k \quad (IH)$$

$$= \sum_i p_i f(a_i)$$