## Today

1) Dominating Sets / The Probabilistic Method

2) Union Bound Failure 1: Overlapping Events

3) Union Bound Failure 2: Independent Events

4) The LLL

# How to Solve Your Fave Randomization Question

To show fact (*)

which RVs/events?

a) Show (*) true if all RVs near $\mathbb{E}$

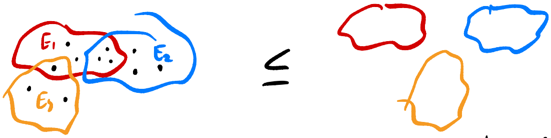
b) Concentration: each RV at $\mathbb{E}$ ($\pm \log n$) w/ good probability

c) Union bound: all RVs " "

Alternatives

## The Union Bound

Given events $E_1, E_2, \ldots$ $\quad P(E_1 \cup E_2 \cup \ldots) \le \sum_i P(E_i)$




$P(E_1 \cup E_2 \cup E_3)$ $\qquad$ $P(E_1) + P(E_2) + P(E_3)$

## Chernoff Bound

Let $X_1, X_2, \ldots, X_n$ be independent RVs s.t. $X_i = \begin{cases} 1 & \text{w/ pr } p_i \\ 0 & \text{o/w} \end{cases}$

Let $X := \sum_i X_i$, $\mu := \mathbb{E}[X]$. Then

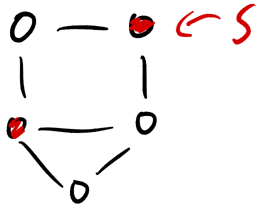
$$\Pr\left(X \ge (1+\delta)\cdot\mu\right) \le \exp\left(-\mu\delta^2/(2+\delta)\right) \quad \forall \delta \ge 0$$

$$\Pr\left(X \le (1-\delta)\cdot\mu\right) \le \exp\left(-\delta^2\mu/2\right) \quad \forall \delta \in (0,1)$$

# Dominating Sets

Given graph $G=(V,E)$, $S\subseteq V$ is a dominating set if $\forall\, u\in V$ we have $\Gamma(u)\cap S\neq\emptyset$

↳ $u\cup\{$neighbors of $u\}$




## Theorem

Every graph of min-degree $d$ has a dominating set of size $\leq O\left(n\cdot\frac{\ln n}{d}\right)$

## Probabilistic Method Approach

1) Randomly make choices → Let $S$ include each vtx. ind. w/ Pr $\frac{16\cdot\ln n}{d}$

2) Show Pr(desired outcome) > 0 (so desired outcome is possible)

Not (necessarily) an efficient alg.

Let $E_S := S \geq \frac{32\cdot\ln n}{d}\cdot n$

For $v\in V$, let

$E_v := \Gamma(v)\cap S=\emptyset$

$$\Pr\left(\overline{E_S}\cap\bigcap_{v\in V}\overline{E_v}\right)>0$$

$$\Updownarrow$$

$$1-\Pr\left(E_S\cup\bigcup_{v\in V}E_v\right)>0$$

$$\Updownarrow$$

$$\Pr\left(E_S\cup\bigcup_{v\in V}E_v\right)<1$$

# A General Chernoff-Union Bound Analysis

Chernoff

Let $X_v := \mathbb{1}[v \in S]$

Fix $v \in V$, let $Y_v := \sum_{u \in \Gamma(v)} X_v$ so $\mathbb{E}[Y_v] \geq 16 \cdot \ln n$

$$\Pr(E_v) \leq \Pr(Y_v \leq 8 \cdot \log n) = \Pr\left(Y_v \leq \left(1 - \underbrace{\tfrac{1}{2}}_{\delta}\right)\mathbb{E}[X_v]\right)$$

$$\leq \exp(-\mathbb{E}[Y_v]/8)$$

$$= \exp(-2 \ln n)$$

$$= n^{-2}$$

Let $X := \sum_{v \in V} X_v$ so $\mathbb{E}[X] = \frac{16 \ln n}{d} \cdot n$

$$\Pr(E_S) = \Pr\left(X \geq \frac{32 \ln n}{d} n\right) = \Pr\left(X \geq (1 + \underbrace{1}_{\delta})\mathbb{E}[X]\right) \leq \exp\left(-\frac{16 \ln n}{3 \cdot d} \cdot n\right) \leq \frac{1}{n}$$

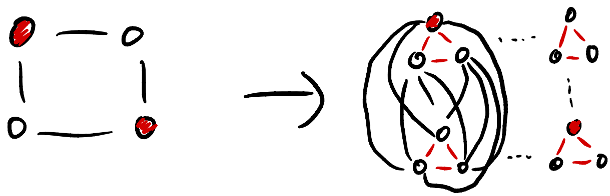
Union

$$\Pr\left(E_S \cup \bigcup_v E_v\right) \underbrace{\leq}_{\text{UB}} \Pr(E_S) + \sum_v \Pr(E_v) \leq \frac{1}{n} + n \cdot n^{-2} = O\left(\frac{1}{n}\right)$$

Suppose "replace" each vertex w/ $K_k$ for some large $k>0$

↳ clique on $k$ nodes

$K_4 \leftrightarrow$ 



$G$ $\qquad\qquad$ $G' = G \bullet K_3$

$n$ nodes $\qquad$ $n \cdot k$ nodes

min deg $d$ $\qquad$ Min deg. $\geq k \cdot d$

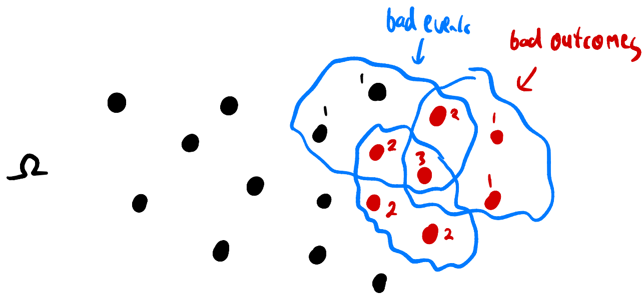
**Claim:** If $G$ has a dominating set of size $\alpha$, so does $G'$ → Let $S$ be $G$'s DS. Choose a one vtx. from each clique in $G'$ corresponding to a vtx. of $S$

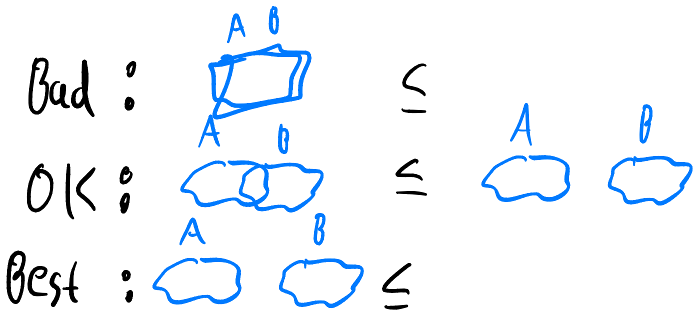
**Claim:** $G'$ has a dom. set of size $\leq O\left(n \cdot \frac{\log n}{d}\right)$

↳ Directly applying Chernoff/UB from before a DS of size $\leq O\left(nk \cdot \frac{\log(n \cdot k)}{d \cdot k}\right) = \omega_k\left(n \cdot \frac{\log n}{d}\right)$

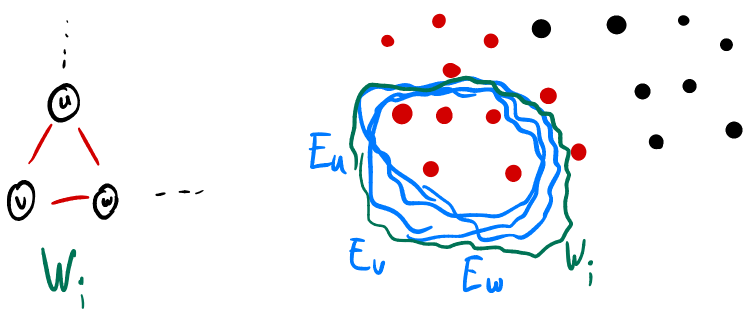
# What Went Wrong?

UB Failure Case 1: bad events highly overlapping




Clique Case:



(Heuristic) Solution: Use non-overlapping bad events
(often means reduce # of bad events)

Clique case: Select each clique w/ Pr $\frac{16 \cdot \ln n}{d}$

$S \leftarrow 1$ node from each selected clique

$E_S :=$ select $\geq n \cdot \frac{32 \ln n}{d}$

For each clique $W_i$ let

$$X_i := |\Gamma(W_i) \cap S|$$

$$E_i := X_i = 0$$

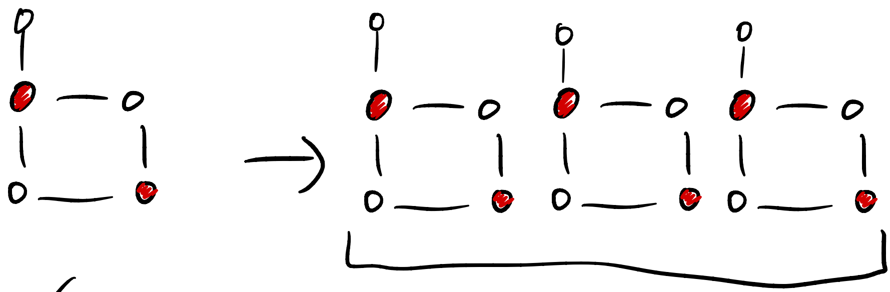
Chernoff (same as before) $\left[ \Pr(E_i) \leq \frac{1}{n^2}, \Pr(E_S) \leq \frac{1}{n} \right.$

Unio. $\left[ \Pr\left(E_S \cup \bigcup_i E_i\right) \underset{\text{UB}}{\leq} \Pr(E_S) + \sum_i \Pr(E_i) \right.$

$$\leq \frac{1}{n} + n \cdot \frac{1}{n^2}$$

$$= O\left(\frac{1}{n}\right)$$

Suppose duplicate graph $\ell$ times for some large $\ell > 0$



$G$

$n$ nodes
Min-degree $d$

$G'$

$n \cdot \ell$ nodes
Min-degree $d$

**Claim:** If $G$ has a dominating set of size $\alpha$, $G'$ has one of size $\leq \ell \cdot \alpha$

↳ Just duplicate the DS

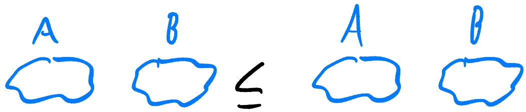
**Claim:** $G'$ has a dominating set of size $\leq O\left(\ell \cdot \frac{n \cdot \ln n}{d}\right)$

↳ But direct Chernoff/UB gives

$$O\left(\ell n \cdot \frac{\ln (n \cdot \ell)}{d}\right) = \omega_{\ell}\left(\ell \cdot \frac{n \ln n}{d}\right)$$

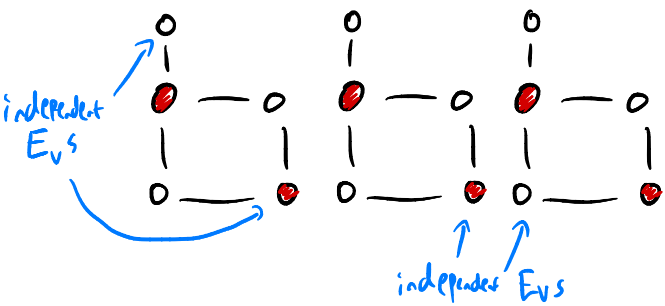
## What Went Wrong?

UB Failure Case 2: events are (mostly) independent

Best : 


↳ When UB is tight, bad events not independent

$$\Pr(A \mid \bar{B}) > \Pr(A)$$




Solution: don't use a UB

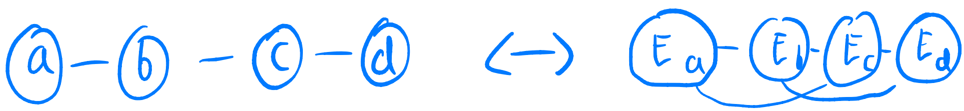
# Lovász Local Lemma (LLL)

Event $A$ is <u>mutually independent</u> of events $\mathcal{B} = \{B_1, B_2, \dots\}$ if $\forall$ partitions $\mathcal{B}_1 \sqcup \mathcal{B}_2 = \mathcal{B}$

$$\Pr(A) = \Pr\Big(A \,\Big|\, \bigcap_{B \in \mathcal{B}_1} B \cap \bigcap_{B \in \mathcal{B}_2} \overline{B}\Big)$$

Given events $\mathcal{A}$, graph $G = (\mathcal{A}, E)$ is a <u>dependency graph</u> if $\forall A \in \mathcal{A}$

$A$ is mutually independent from $\mathcal{A} \setminus \Gamma(A)$




DS Graph

Dep. Graph

<u>Symmetric LLL</u>: Given events $\mathcal{A}$ w/ dependency graph $G$, if $\exists\, P, \Delta$ s.t.

1) $\Pr(A) \leq P \quad \forall A \in \mathcal{A}$

2) $\text{max-deg}(G) \leq \Delta$

3) $e \cdot P \cdot \Delta \leq 1$

If DG a clique says same thing as UB (up to constants)

then $\Pr\Big(\bigcap_{A \in \mathcal{A}} \overline{A}\Big) > 0$

Like a UB on each events neighborhood

(Also $\exists$ poly-time alg. to find outcome $\in \bigcap_{A \in \mathcal{A}} \overline{A}$)

# Theorem

Every graph of min-degree $d$ has a dominating set of size $\le O\left(n\cdot\frac{\ln d}{d}\right)$ $\left(< O\left(n\frac{\ln n}{d}\right)\right)$ (for $d \ge 100$)

Each $u \in S$ w/ Pr $\frac{32 \cdot \ln d}{d}$ (independently)

For $v \in V$, let
$E_v := \Gamma(v) \cap S = \emptyset$

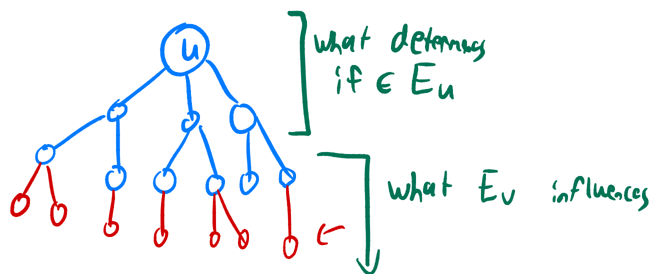
Chernoff $\left[\ \Pr(E_v) \le \frac{1}{d^4}\right.$

LLL

Dependency graph

Let $\Gamma^{(2)}(u) := \bigcup_{v \in \Gamma(u)} \Gamma(v)$ so $|\Gamma(u)| \le 1+d+d^2$

$E_u$ is MI from $\{E_v : v \in V \setminus \Gamma^{(2)}(u)\}$




So DG has edge $\{E_u, E_v\}$ iff $v \in \Gamma^{(2)}(u)$

So $e \cdot p \cdot \Delta = e \cdot \frac{1}{d^4} \cdot (1+d+d^2) \le 1$

A Subtle Bug: didn't guarantee $|S|$ is small

Let $E_S := |S| > 128 n \cdot \frac{\ln \cdot d}{d}$

So want to avoid $E_S \cup \bigcup_{v \in V} E_v$

Problem: $E_S$ not clearly MI from any other bad events

↳ $\Delta$ as large as $n$

<u>Asymmetric LLL</u>: Given events $\mathcal{A}$ w/ dependency graph $G$,

if $\exists\ \Delta, \{x_A : A \in \mathcal{A}\}$

$$\Pr(A) \le x_A \cdot \prod_{B \in \Gamma_G(A)} (1 - x_B) \quad \forall A \in \mathcal{A}$$

then $\Pr\left(\bigcap_{A \in \mathcal{A}} \bar{A}\right) > 0$

$$\Pr(E_S) \le \exp(-2n)$$

Let $x_{E_S} = \exp(-n)$ and $x_{E_v} = \frac{2}{d^4}$

So $x_{E_S} \prod_v (1 - x_v) = \exp(-n) \cdot \underbrace{\left(1 - \frac{2}{d^4}\right)^n}_{\ge \left(1 - .63 \frac{4}{d^4}\right)^n \ge \exp(-n)} \ge \exp(-2n) \ge \Pr(E_S)$

And $\forall v \in V\ x_{E_v} \prod_{u \in \Gamma_v^{(4)}(v)} (1 - x_{E_u}) \ge \frac{2}{d^4}\left(1 - \frac{2}{d^4}\right)^{d^2} \ge \frac{2}{d^4} \cdot \exp(-1/d^2) \ge \frac{1}{d^4} \ge \Pr(E_u)$