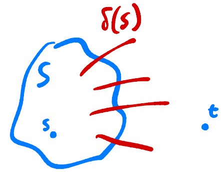
Today

- Global Min Cut
- Sparsification Framework
- Gomory-Hu Trees
- Key Cut Lemma
- GHT Alg. + Analysis

Recall

Given $G=(V,E,w)$ and $s,t \in V$, an s-t cut is $S \subseteq V$ s.t. $s \in S$, $t \notin S$

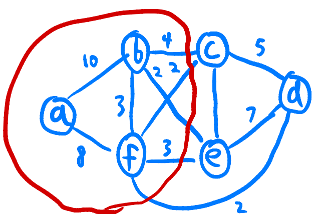


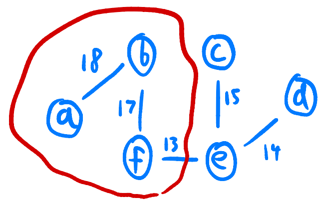
$w(\delta(S)) := \sum_{e \in \delta(S)} w(e)$

S is a min cut if $w(\delta(S))$ is minimum among all s-t cuts

E.g.

(from Vazirani)



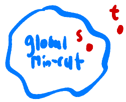


Given graph $G=(V,E,w)$ the <u>global min-cut</u> is the non-empty $S \subset V$ minimizing $w(\delta(S))$

Naive algorithm: return S s.t. $w(\delta(S)) = \min_{s,t \in V} MC_{st}(G)$

Takes $O(n^2)$ calls to s-t MC

Correct b/c eventually guess



<u>Sparsification Framework</u> (to solve graph problem on G)

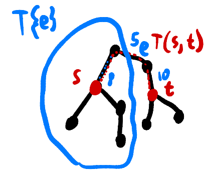
α) Find simple graph H (approximately) preserving structure of problem on G → Today: Gomory-Hu trees for global min-cut

β) Solve problem on H

γ) Convert solution on H to solution on G

<u>Min-st Cuts on Trees</u>

For tree T, let $T(s,t)$ be the unique (simple) $s \rightsquigarrow t$ path in T and let $T\{e\}$ be cut s.t. $\delta(T\{e\}) = \{e\}$ (will use convenient side)



<u>Observe</u>: if $T=(V,E_T,w_T)$ is a tree then $\forall s,t \in V$, $S \subseteq V$ is an s-t MC iff $S = T\{e\}$ for some $e \in E$ s.t. $w(e) = \min_{\hat{e} \in T(s,t)} w(\hat{e})$

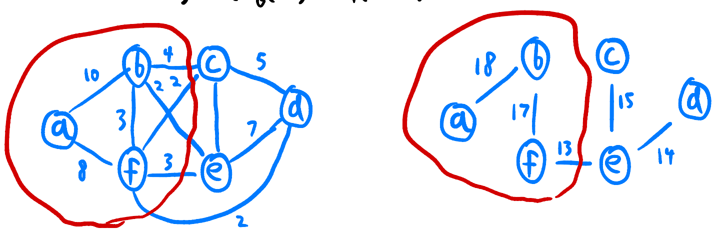
<u>Gomory-Hu Trees</u>

Given $G=(V,E,w)$, a <u>Gomory-Hu tree</u> is a tree $T=(V,E_T,w_T)$ s.t. $\forall s,t \in V$ if $W \subseteq V$ is an s-t MC in T then

1) W is an s-t MC in G
2) $w(\delta_G(S)) = w_T(\delta_T(S))$

E.g.

(from Vazirani)



<u>Theorem</u>: every edge-weighted graph has a GH-tree computable w/ $O(n)$ calls to s-t MC

<u>Corollary</u>: Can solve global min-cut w/ $O(n)$ calls to s-t MC

Compute a GH tree $T=(V,E_T,w_T)$ (α)

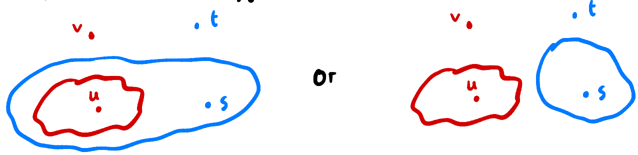
Let $e^* := \operatorname{argmin}_{e \in E_T} w_T(e)$ (β) and return $T\{e^*\}$ (γ)

Correct b/c $\forall s,t$ if $e_{st} := \operatorname{argmin}_{e \in T(s,t)} w(e)$ then $T\{e_{st}\}$ is an s-t MC in G

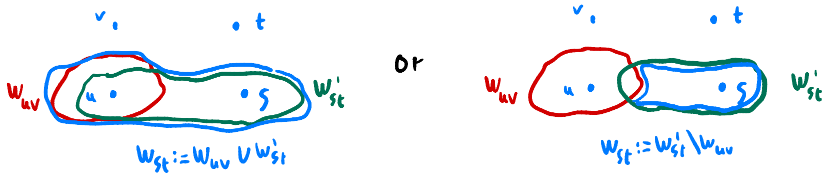
Runtime: $O(n)$ calls for GHT

Key Cut Lemma: Suppose W_{uv} is a u-v MC for $u,v \in V$ and $s,t \notin W_{uv}$.

Then $\exists$ an s-t MC W_{st} s.t. $W_{uv} \subseteq W_{st}$ or $W_{uv} \cap W_{st} = \emptyset$



Intuition: let W'_{st} be an arbitrary s-t MC



Skip past this

Given a finite set V, $f: 2^V \to \mathbb{R}$ is <u>submodular</u> if $\forall A,B \subseteq E$

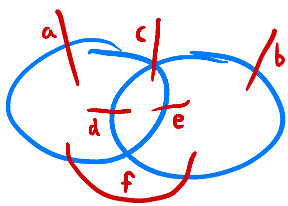
$$f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$$

Given a finite set V, $f: 2^V \to \mathbb{R}$ is <u>Posimodular</u> if $\forall A,B \subseteq E$

$$f(A \setminus B) + f(B \setminus A) \leq f(A) + f(B)$$

Lemma: Given graph $G=(V,E,w)$, $f(V) := w(\delta(V))$ is submodular and posimodular

Partition $\delta(V)$ as



$\leftarrow$ letter corresponds to w of outgoing edges

Have $f(A)+f(B) = (a+c+e+f)+(b+c+d+f) = a+b+2c+d+e+2f$

$f(A \cup B) + f(A \cap B) = (a+c+b)+(c+d+e) \quad = a+b+2c+d+e \leq$

$f(A \setminus B) + f(B \setminus A) = (a+d+f)+(b+e+f) \quad = a+b+e+2f \leq$

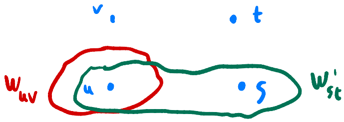
Claim: If $a+b=c+d$ and $a \geq c$, $b \geq d$ then $a=c$, $b=d$ for $a,b,c,d \in \mathbb{R}$ b/c $a>c$ or $b>d \to a+b>c+d$

Proof of Key Cut Lemma

Let W'_{st} be an arbitrary s-t MC

WLOG suppose $s \in W'_{st}$, $u \in W_{uv}$ and $W'_{st} \cap W_{uv} \neq \emptyset$ but $W_{uv} \not\subseteq W'_{st}$

Suppose $u \in W'_{st}$



Then $W_{uv} \cap W'_{st}$ is a u-v cut and $W_{uv} \cup W'_{st}$ is an s-t cut

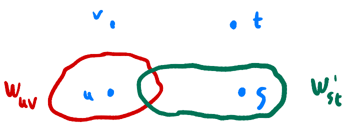
So $f(W_{uv} \cap W'_{st}) \geq f(W_{uv})$ and $f(W_{uv} \cup W'_{st}) \geq f(W'_{st})$

$$f(W_{uv}) + f(W'_{st}) \underset{\text{Submodularity}}{\geq} f(W_{uv} \cap W'_{st}) + f(W_{uv} \cup W'_{st}) \underset{W_{uv}\ W_{st}\ \text{MCs}}{\geq} f(W_{uv}) + f(W'_{st})$$

so $f(W_{uv} \cap W'_{st}) + f(W_{uv} \cup W'_{st}) = f(W_{uv}) + f(W'_{st})$ so $f(W'_{st}) = f(W'_{st} \cup W_{uv})$ by claim

so $W_{st} := W_{uv} \cup W'_{st}$ is an s-t MC s.t. $W_{uv} \subseteq W_{st}$

Suppose $u \notin W'_{st}$



Then $W_{uv} \setminus W'_{st}$ is a u-v cut and $W'_{st} \setminus W_{uv}$ is an s-t cut

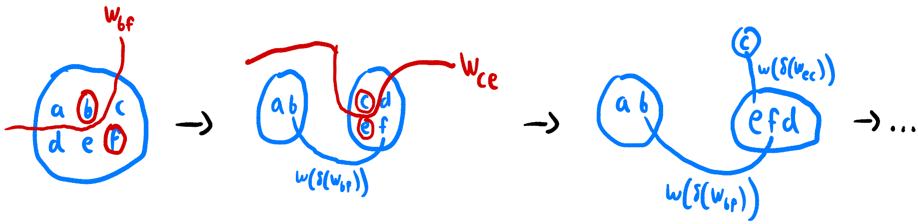
So $f(W_{uv} \setminus W'_{st}) \geq f(W_{uv})$ and $f(W'_{st} \setminus W_{uv}) \geq f(W'_{st})$

$$f(W_{uv}) + f(W'_{st}) \underset{\text{Posimodularity}}{\geq} f(W_{uv} \setminus W'_{st}) + f(W'_{st} \setminus W_{uv}) \underset{W_{uv}\ W_{st}\ \text{MCs}}{\geq} f(W_{uv}) + f(W'_{st})$$

so $f(W_{uv} \setminus W'_{st}) + f(W'_{st} \setminus W_{uv}) = f(W_{uv}) + f(W'_{st})$ so $f(W'_{st}) = f(W'_{st} \setminus W_{uv})$ by claim

so $W_{st} := W'_{st} \setminus W_{uv}$ is an s-t MC s.t. $W_{uv} \cap W_{st} = \emptyset$

Algorithm Intuition: repeatedely split up vertices by s-t MCs

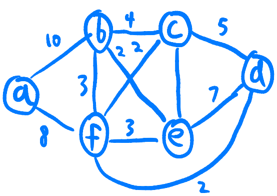


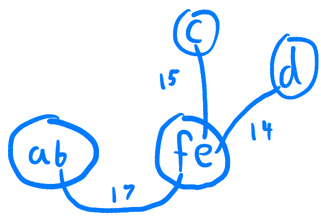
Given $G=(V,E,w)$ a Partial Gomory-Hu Tree (My terminology) consists of a partition $\mathcal{C}$ of V, a tree $T=(\mathcal{C},E_T,w_T)$ where $e=(C_1,C_2)\in E_T$ "corresponds to" some $s\in C_1$, $t\in C_2$ and if s,t correspond to $e\in E_T$ then if $W_{st}:=\bigcup_{C\in T\{e\}} C$

1) W_{st} is an s-t MC in G

2) $w(\delta(W_{st}))=w_T(e)$

E.g.





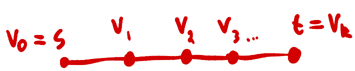
Claim: If $T=(\mathcal{C},E_T,w_T)$ is a partial GHT of $G=(V,E,w)$ w/ $\mathcal{C}=\{\{v\}:v\in V\}$ then T is a GHT of G

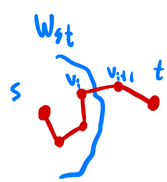
Fix $s,t\in V$; suffices to show $\forall e\in T(s,t)$, $w(e)\geq MC_{st}(G)$ and $\exists e\in T(s,t)$ s.t. $w(e)=MC_{st}(G)$ (by observation)

$w(e)\geq MC_{st}(G)$ $\forall e\in T(s,t)$ b/c $T\{e\}$ is an s-t cut $\forall e\in T(s,t)$ and $w(e)=w(\delta(T\{e\}))$

$\exists e\in T(s,t)$ s.t. $w(e)=MC_{st}(G)$

Label vertices of $T(s,t)$ as





Consider some s-t MC W_{st}

$\exists$ i s.t. $v_i\in W_{st}$ but $v_{i+1}\notin W_{st}$; Let $e=\{v_i,v_{i+1}\}$ and $W:=T\{v_i\}$

W_{st} is a v_i-v_{i+1} cut and $T\{e\}$ is an s-t cut

So $MC_{v_iv_{i+1}}(G)\leq w(\delta(W_{st}))=MC_{st}(G)\leq MC_{v_iv_{i+1}}(G)$

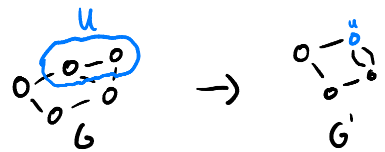
So $w(e)=MC_{v_iv_{i+1}}(G)=MC_{st}(G)$ as required

Want all of previous MCs on one side of new MC

Cut lemma helps w/ this but need trick to force algorithm to do this

Given $G=(V,E,w)$, graph $G'=(V',E',w')$ is the result of **contracting $U\subseteq V$**

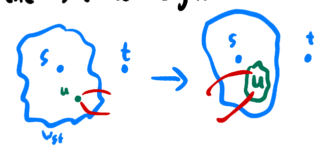
if $V'=V\setminus U+u$ (New vertex) $\quad E':=E\setminus\delta(U)+\{\{u,v\}:\{x,v\}\in\delta(U), x\in U\}$, $\quad w'(\{x,y\}) = \begin{cases} w(\{x,y\}) & \text{if } x\neq u \\ w(\{\cdot,y\}) & \text{if } x=u \end{cases}$



Observe: if $s,t\notin U$ then contracting U can only increase the s-t MC weight

Let G_U be G w/ U contracted

If W_{st} is an s-t MC in G_U then (after uncontracting U) it is also an s-t MC in G of equal weight



Notation abuse: $U_i := \bigcup_{C\in U_i} C$

GH-Tree Alg.

$\mathcal{C} \leftarrow \{V\}$

While $\exists C\in\mathcal{C}$ s.t. $|C|\geq 2$

$\quad T \leftarrow \text{Breakup}(T,C)$

Return T

Breakup(T, C)

Let $s,t\in C$ be distinct but arbitrary

Let $U_1, U_2, \ldots$ be connected components of $T-C$

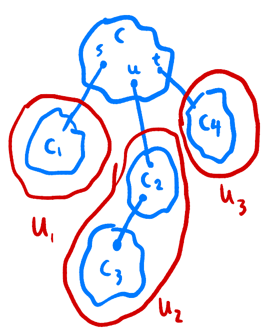
Let G_C be G w/ each $U_1, U_2, \ldots$ contracted

Let W be an s-t MC in G_C

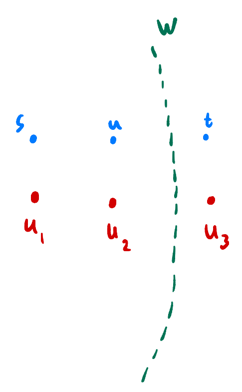
Replace $C\in\mathcal{C}$ with $W\cap C$ and $C\setminus W$

Add edge $\{W\cap C, C\setminus W\}$ of weight $w(\delta(W))$

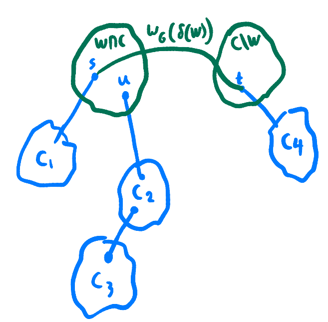
w/ corresponding vertices s, t



T



G_C



Breakup(T, C)

Proof of Theorem

Runtime

Done once T has n nodes; each breakup adds 1 node to T using 1 s-t MC call

↳ Note: Can also efficiently maintain all G_c, so really gives an efficient algorithm

Correctness

Since alg. terminates, by claim suffices to show

if T is a partial GH tree then so is breakup(T, C)

Suffices to show $MC_{st}(G_c) = MC_{st}(G)$

$MC_{st}(G) \le MC_{st}(G_c)$ by observation

$MC_{st}(G_c) \le MC_{st}(G)$

Let G_i be the result of contracting $U_1, U_2, \dots, U_i$

By induction, suffices to show $MC_{st}(G_i) \le MC_{st}(G_{i-1})$ $\forall i$

U_i is a u-v MC in G not containing s, t for some u, v

So by key cut lemma, $\exists$ an s-t MC W_{st} in G_{i-1} s.t. $U_i \subseteq W_{st}$ or $U_i \cap W_{st} = \emptyset$

W_{st} is also an s-t cut in G_i, showing $MC_{st}(G_i) \le MC_{st}(G_{i-1})$

E.g.