Today

1) Probability Review

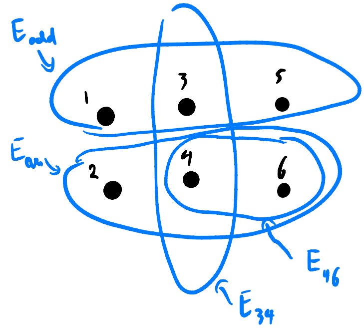
2) Fun w/ Gaussians

(Discrete) Probability Review

A **discrete probability space** consists of

1) A (countable) set of "outcomes" Ω called the sample space
2) A **probability function** $P : \Omega \to [0,1]$

$$\sum_{o \in \Omega} P(o) = 1$$



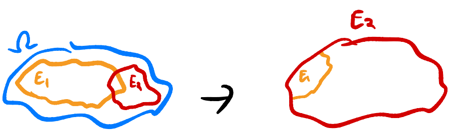
$P(o) = 1/6 \quad \forall\, o \in \Omega$

An **event** E is a subset of Ω

The probability of E is $P(E) := \sum_{o \in E} P(o)$

$E_1 \cup E_2 \leftrightarrow$ "E_1 or E_2"
$E_1 \cap E_2 \leftrightarrow$ "E_1 and E_2"

The probability of **E_1 conditioned on E_2** is $P(E_1 \mid E_2) := \dfrac{P(E_1 \cap E_2)}{P(E_2)}$



E.g. $P(E_{46} \mid E_{even}) = \frac{2}{3}$ and $P(E_{34} \mid E_{even}) = \frac{1}{3}$

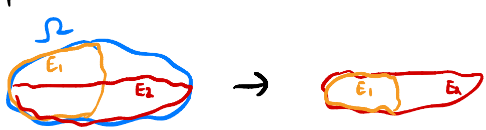
Events E_1 and E_2 are **independent** if $P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$

$$\Updownarrow$$

$$P(E_1 \mid E_2) \cdot P(E_2) = P(E_1) \cdot P(E_2)$$

$$\Updownarrow$$

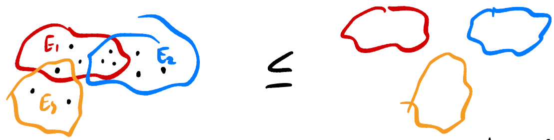
$$P(E_1 \mid E_2) = P(E_1)$$



E_{34}, E_{even} independent

The Union Bound

Thm: Given "bad" events $E_1, E_2, \dots, E_k$ $\quad P(E_1 \cup E_2 \cup \dots \cup E_k) \leq \sum_{i=1}^{k} P(E_i)$



$P(E_1 \cup E_2 \cup E_3) \qquad P(E_1) + P(E_2) + P(E_3)$

Proof: Let $\quad n_o := \left| \{ i : E_i \cap o \neq \emptyset \} \right|$

Then $\quad P(E_1 \cup E_2 \cup \dots) = \sum_{o \in E_1 \cup E_2 \cup \dots \cup E_k} P(o)$

$$\leq \sum_{o \in E_1 \cup E_2 \cup \dots \cup E_k} P(o) \cdot n_o$$

$$= \sum_{o} \sum_{i : o \in E_i} P(o)$$

$$= \sum_{i=1}^{k} \sum_{o \in E_i} P(o) \quad \text{(switch order of summation)}$$

$$= \sum_{i=1}^{k} P(E_i)$$

Random Variables

A **discrete RV** is a function $X : \Omega \to \mathbb{R}$

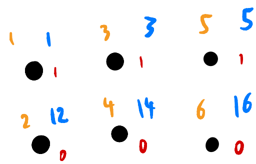
"$X = a$" is event $\{ o \in \Omega : X(o) = a \}$

Random variables X, Y are **independent** if $\Pr(X = a \cap Y = b) = \Pr(X = a) \cdot \Pr(Y = b) \ \forall a, b \in \mathbb{R}$

$X :=$ dice #

$Y := \mathbb{1}(\text{dice is odd})$

$Z := X + 10(1 - Y)$



Expectation of RV X is $\mathbb{E}[X] := \sum_i i \cdot \Pr(X = i)$ → E.g. $\mathbb{E}[Y] = \frac{51}{6}$

Conditional Expectation of RV X conditioned on $Y = y$ is

$$\mathbb{E}[X \mid Y = y] := \sum_i i \cdot \Pr(X = i \mid Y = y)$$

Variance of RV X is $\mathrm{Var}(X) := \mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$ ($\sigma(X) = \sqrt{\mathrm{Var}(X)}$ is standard deviation)

Covariance of RVs X, Y is $\mathrm{Cov}(X, Y) := \mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right]$

Common Identities (simple calculations)

$$\mathbb{E}[aX+bY] = a\,\mathbb{E}[X] + b\,\mathbb{E}[Y]$$

$$\text{Var}(X) = \mathbb{E}[X^2] - \left(\mathbb{E}[X]\right)^2$$

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\,\text{Cov}(X,Y)$$

$$\text{Var}(cX) = c^2 \cdot \text{Var}(X)$$

If X, Y independent

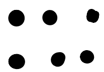
$$\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$$

$$\text{Cov}(X,Y) = 0$$

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$$

Discrete RV

Ω_S



or

$\mathbb{Z}$

$\Pr(X=a)$

$\mathbb{E}[X] = \sum_i i \cdot \Pr(X=i)$

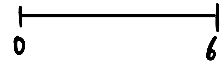
$\Pr(X \le a) = \sum_{i \le a} \Pr(X=i)$

$\sum_i \Pr(X=i) = 1$

Events $= 2^{\Omega}$

Probability Function

Continuous RV



or

$\mathbb{R}$

Ω_S

pdf $\varphi(a)$

$\mathbb{E}[X] := \int_i i \cdot \varphi(i)\, di$

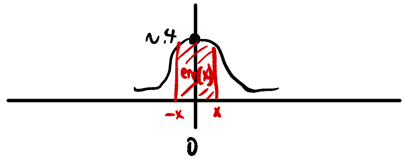
CDF $\Phi(a) := \int_{-\infty}^{a} \varphi(i)\, di$

$\int_{-\infty}^{\infty} \varphi(i)\, di = 1$

σ-algebra $\subseteq 2^{\Omega}$

Measure

Gaussians



$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$Z \sim N(0,1)$ → "Standard Gaussian"

$\mathbb{E}[X] = 0$ by symmetry around y-axis

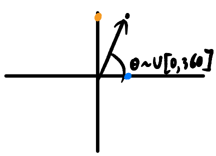
Amazing Fact 1: $\int_{-\infty}^{\infty} \varphi(x) = 1$ (no anti-deriv. w/ ele. fns.)

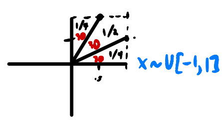
$$\text{erf}(x) = \frac{1}{\sqrt{\pi}} \int_{-x}^{x} e^{-t^2}$$

Can derive $\text{var}(X) = 1$ from this (hence "$N(0,1)$")

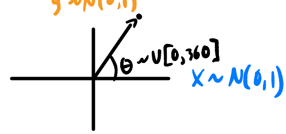
Amazing Fact 2: Rotational symmetry

Suppose want uniformly random direction in $\mathbb{R}^2$ w/ independent coordinates





Uniform = bad

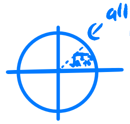


Gaussian = good

Fix (a,b) w/ length $\sqrt{a^2+b^2}$

$$\Pr(V=(a,b)) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(a^2+b^2)}$$

The same probability $\forall\ u=(x,y)$ s.t. $\sqrt{x^2+y^2} = \sqrt{a^2+b^2}$



all equally likely

Generalizes to $\mathbb{R}^k$ for $k>2$

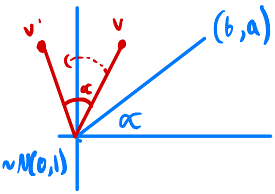
Dfn. $N(\mu, \sigma^2) = \mu + \sigma Z$ where $Z \sim N(0,1)$ is a "non-Standard Gaussian"

If $X \sim N(\mu, \sigma^2)$ then $\mathbb{E}[X] = \mu$ and $\mathrm{Var}(X) = \sigma^2$ $\rightarrow$ Prove

Amazing Fact 3: Sum of Gaussians is a Gaussian

If $X \sim N(0,1)$, $Y \sim N(0,1)$, X, Y independent, $a, b \in \mathbb{R}$
then $aX + bY \sim N(0, a^2 + b^2)$

Let $L = \sqrt{a^2 + b^2}$



$(b,a) = (L \cdot \cos(\alpha), L \cdot \sin(\alpha))$

$\Updownarrow$

$\sin \alpha = \frac{a}{L}$

$\cos \alpha = \frac{b}{L}$

Consider random vector v'
1) $v \sim (X, Y)$ where $X, Y \sim N(0,1)$
2) $v' \leftarrow v$ rotated α degrees

v' has same distribution as v (by rot. symm.)

v' has coordinates $(\sim, \underbrace{X \cdot \sin\alpha + Y \cdot \cos\alpha}_{\sim N(0,1)})$ (by trig.)

$\rightarrow \frac{a}{L} \cdot X + \frac{b}{L} \cdot Y \sim N(0,1)$

$\rightarrow aX + bY \sim L \cdot N(0,1) = N(0, a^2 + b^2)$

Amazing Fact 4: Central Limit Theorem

Suppose $X_1, X_2, \ldots X_n$ are i.i.d. w/ $\mathbb{E}[X_i] = \mu$, $\mathrm{Var}(X_i) = \sigma^2$

Let $S_n = \sum_i X_i$ (so $\mathbb{E}[S_n] = \mu \cdot n$ and $\mathrm{Var}(S_n) = n \cdot \sigma^2$)

Let $\hat{S}_n = \frac{S_n - n \cdot \mu}{\sqrt{n} \cdot \sigma}$ (so $\mathbb{E}[\hat{S}_n] = 0$ and $\mathrm{Var}(\hat{S}_n) = 1$)

Then $\hat{S}_n \rightarrow Y$ where $Y \sim N(0,1)$

$\Updownarrow$

$\Pr(\hat{S}_n \leq a) = \Pr(Y \leq a) \pm o_n(1) \quad \forall a \in \mathbb{R}$