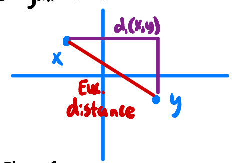
Today

- Sparsest Cut
- Reduction to H-Sparsest Cut
- $O(\log n)$-approximation for H-Sparsest Cut via Bourgain + Cut cone
- Expanders

Recall



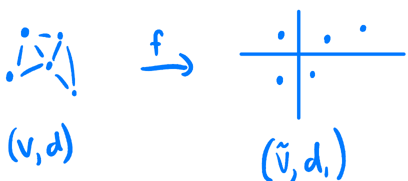
Given $x, y \in \mathbb{R}^n$ define $d_1(x,y) := \sum_i |x_i - y_i|$

Bourgain's Theorem: given any n-point metric (V, δ), $\exists$ (Poly-time computable) embedding f w/ distortion $O(\log n)$ of (V, δ) into $(\tilde{V}, d_1)$ for $\tilde{V} \subseteq \mathbb{R}^{O(\log^2 n)}$

1-1 Correspondance Between Metrics and Vectors

$$(V, d) \text{ s.t. } V = \{v_1, v_2, \dots, v_n\} \quad \longleftrightarrow \quad \begin{pmatrix} d(v_1, v_2) \\ d(v_1, v_3) \\ \vdots \\ d(v_1, v_n) \\ d(v_2, v_3) \\ \vdots \end{pmatrix} \in \mathbb{R}^{\binom{n}{2}}$$

(V, δ) is an **ℓ_1 metric** if it embeds isometrically into $(\hat{V}, d_1)$ for $\hat{V} \subseteq \mathbb{R}^n$ for some n

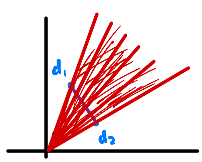


(V, d_S) is a **cut metric** for $S \subseteq V$ s.t.

$$d_S(u,v) = \begin{cases} 0 & u,v \in S \text{ or } u,v \notin S \\ 1 & \text{otherwise} \end{cases}$$

Let $\ell_1(V) \subseteq \mathbb{R}^{\binom{n}{2}}$ be all ℓ_1 metrics on V and let $\mathrm{Cut}(V) \subseteq \mathbb{R}^{\binom{n}{2}}$ be all cut metrics of V

The **convex cone** of $D \subseteq \mathbb{R}^{\binom{n}{2}}$ is $\mathrm{Cone}(D) := \left\{ \sum_{d \in D} \alpha_d \cdot d : \alpha_d \geq 0 \ \forall d \right\}$



Theorem: $\ell_1(V) = \mathrm{Cone}(\mathrm{Cut}(V)) \ \forall V.$

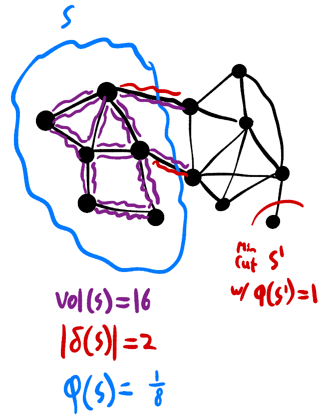
Also, given $d \in \ell_1(V)$ + ℓ_1 embedding can poly-time compute $d_i \in \mathrm{Cut}(V)$, $\alpha_i \in \mathbb{R} \ \forall i$ s.t. $d = \sum_i \alpha_i \cdot d_i$

Sparsest Cut

Given connected graph $G=(V,E)$

The <u>volume</u> of $S \subseteq V$ is $\mathrm{Vol}(S) := \sum_{u \in S} \deg(u)$

The <u>conductance</u> of $S \subseteq V$ is $\varphi(S) := \dfrac{|\delta(S)|}{\min(\mathrm{vol}(S), \mathrm{vol}(V \setminus S))}$



<u>Sparsest Cut Problem:</u> find $S \subseteq V$ minimizing $\varphi(S)$

Note: not always $\min_{s,t}$ min-cut(s,t)

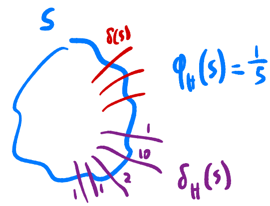
Note: $\varphi(S) \in (0,1]$ always b/c $\mathrm{vol}(S) \geq |\delta(S)|$

Sparsest Cut is NP-hard

H-Sparsest Cut

Also given complete edge-weighted graph $H=(V, E_H, w)$ $\quad w: E \to \mathbb{R}_{\geq 0}$

The <u>H-conductance</u> of $S \subseteq V$ is $\varphi_H(S) := \dfrac{|\delta_G(S)|}{w(\delta_H(S))}$ $\quad w(\delta_H(S)) = \sum_{e \in \delta_H(S)} w(e)$



<u>H-Sparsest Cut Problem:</u> find $S \subseteq V$ minimizing $\varphi_H(S)$

<u>Theorem:</u> $\exists$ poly-time $O(\log n)$-approximation for H-Sparsest Cut

<u>Corollary:</u> $\exists$ poly-time $O(\log n)$-approximation for Sparsest Cut

Let $w_H(\{u,v\}) = \dfrac{\deg_G(u) \cdot \deg_G(v)}{m}$ $\forall u,v$ and let $V_S = \mathrm{Vol}(S)$, $\bar{V}_S = \mathrm{Vol}(V \setminus S)$

so $w(\delta_H(S)) \underset{\text{H complete}}{=} \sum_{\substack{u \in S \\ v \notin S}} \dfrac{\deg_G(u) \cdot \deg_G(v)}{m} = \dfrac{1}{m} \sum_{u \in S} \deg_G(u) \sum_{v \notin S} \deg_G(v) = \dfrac{1}{m} \sum_{u \in S} \deg_G(u) \cdot \bar{V}_S = \dfrac{1}{m} V_S \cdot \bar{V}_S$

$\forall S \subseteq V \quad \frac{1}{2} \cdot \varphi(S) \leq \varphi_H(S) \leq \varphi(S)$ → Corollary follows b/c objective changed by ≤ 2

$$\frac{m}{2} \cdot \min(V_S, \bar{V}_S) \leq \min(V_S, \bar{V}_S) \cdot \max(V_S, \bar{V}_S) = V_S \cdot \bar{V}_S \leq m \cdot \min(V_S, \bar{V}_S)$$

So $w(\delta_H(S)) \in \left[\frac{1}{2} \cdot \min(V_S, \bar{V}_S), \min(V_S, \bar{V}_S)\right]$

so $\forall S \subseteq V \quad \frac{1}{2} \cdot \varphi(S) \leq \varphi_H(S) \leq \varphi(S)$

H-Sparsest Cut via Cut Metrics

Variable $d(u,v)$ for each $u,v \in V$; let $d(e) := d(u,v)$ for $e = \{u,v\}$

Not an LP n̈

$$\min_{d} \frac{\sum'_{e\in E} d(e)}{\sum'_{u,v} d(u,v)\cdot w(\{u,v\})} \quad \text{s.t.} \quad d \text{ is a cut metric on } V \qquad (1)$$

$\downarrow$ Leighton - Rao Relaxation

"d is a metric"

$$\min_{d} \frac{\sum'_{e\in E} d(e)}{\sum'_{u,v} d(u,v)\cdot w(\{u,v\})} \quad \text{s.t.} \quad \underbrace{\begin{aligned} &d(x,x)=0 \quad \forall x\\ &d(x,y)=d(y,x) \quad \forall x,y\\ &d(x,y)\le d(x,z)+d(z,y) \quad \forall x,y,z\end{aligned}}_{\text{"d is a metric"}} \qquad (2)$$

$\updownarrow$

An LP ü

$$\min_{d} \sum'_{e\in E} d(e) \quad \text{s.t.} \quad \begin{aligned} &d(x,x)=0 \quad \forall x\\ &d(x,y)=d(y,x) \quad \forall x,y\\ &d(x,y)\le d(x,z)+d(z,y) \quad \forall x,y,z\\ &\sum'_{u,v} d(u,v)\cdot w(\{u,v\}) = 1\end{aligned} \qquad (3)$$

Let OPT_i be optimal of (i) above and O_i be objective of (i)

Claim: $OPT_3 \le OPT$

$OPT_2 \underset{\substack{\text{min over}\\ \text{larger set}}}{\le} OPT_1 \underset{\substack{\text{usual}\\ \text{1-1 corr.}}}{=} OPT$

$OPT_3 = OPT_2$

$OPT_3 \le OPT_2$: Suppose d_2 optimal for (2) and let $\alpha = \sum'_{u,v} d_2(u,v)\cdot w(\{u,v\})$ so $O_2(d_2) = \frac{O_3(d_2)}{\alpha}$; $OPT_2 = O_2(d_2) = \frac{O_3(d_2)}{\alpha}$

But $\frac{1}{\alpha}\cdot d_2$ is feasible for (3) so $OPT_3 \le O_3(\frac{1}{\alpha}\cdot d_2) = \frac{O_3(d_2)}{\alpha} = OPT_2$

$OPT_2 \le OPT_3$: Suppose d_3 optimal for (3) so $OPT_3 = O_3(d_3)$

But d_3 feasible for (2) so $OPT_2 \le O_2(d_3) \overset{\text{extra constraint of (3)}}{=} O_3(d_3) = OPT_3$

Proof of Theorem

Idea: Solution to (3) $\overset{\text{Bourgain}}{\rightsquigarrow}$ ℓ_1 metric $\overset{\text{cut cone}}{\rightsquigarrow}$ Cut on G

lose $O(\log n)$

Alg

- Let d be an optimal solution to (3)
- Let $(\tilde{V}, d_1)$ be what Bourgain embeds (V,d) into w/ $O(\log n)$ distortion
- $d_1 \in \mathrm{Cone}(\mathrm{Cut}(\tilde{V}))$ so $d_1 = \sum_{S \in \mathcal{S}} \alpha_S \cdot d_S$ s.t. $\alpha_S \geq 0$ and d_S a cut metric w/ cut S
- Let S^* be $S \in \mathcal{S}$ minimizing $\varphi_H(S)$
- Return S^*

Poly-time b/c Bourgain + Cut Cone poly-time doable

To see approximation guarantee:

For $S \in \mathcal{S}$ let $N_S := \sum_e d_S(e)$ and $D_S := \sum_{u,v} d_S(u,v) \cdot w(u,v)$

$$\text{So} \quad S^* = \underset{S \in \mathcal{S}}{\operatorname{argmin}} \frac{N_S}{D_S} = \underset{S \in \mathcal{S}}{\operatorname{argmin}} \frac{\alpha_S \cdot N_S}{\alpha_S \cdot D_S}$$

$$\varphi_H(S^*) = \frac{N_{S^*}}{D_{S^*}} = \frac{\alpha_{S^*} \cdot N_{S^*}}{\alpha_{S^*} \cdot D_{S^*}} \leq \frac{\sum_S \alpha_S \cdot N_S}{\sum_S \alpha_S \cdot D_S} = \frac{\sum_e \sum_S \alpha_S d_S(e)}{\sum_{u,v} w(u,v) \sum_S \alpha_S \cdot d_S(u,v)} = \frac{\sum_e d_1(e)}{\sum_{u,v} w(u,v) \cdot d_1(u,v)}$$

$$\min_i \frac{a_i}{b_i} \leq \frac{\sum_i a_i}{\sum_i b_i}$$

$$\leq O(\log n) \cdot \frac{\sum_e d(e)}{\sum_{u,v} d(u,v) \cdot w(u,v)} = O(\log n) \cdot OPT_3 \leq O(\log n) \cdot OPT$$

Bourgain has $O(\log n)$ distortion

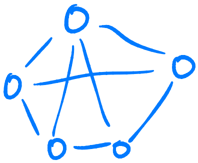
Claim

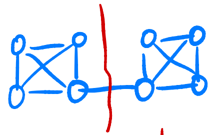
Expanders

$G=(V,E)$ is a φ-edge-expander if $\varphi(S) \geq \varphi$ $\forall$ non-empty $S \subset V$

φ-expander for short

E.g.





$\varphi(S) = \frac{1}{\Omega(n^2)}$

① K_n is an $\Omega(1)$-expander

Complete graph on n vertices

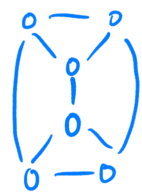
② Above is a $O(\frac{1}{n^2})$-expander

Among most well-studied graph classes in TCS. Why?

Useful for impossibility results

Fact: $\forall n_0$, $\exists$ an $n \geq n_0$-node 3-regular $\Omega(1)$-expander

$\deg(v)=3$ $\forall v$



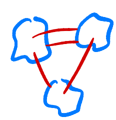
$\exists$ sparse graphs that are as connected as K_n

Shows:

- LDD separation quality $\Omega(\frac{\log n}{\Delta})$
- Embeddings into ℓ_1 or ℓ_2 require $\Omega(\log n)$ distortion (tightness of Bourgain)
- "Probabalistic tree embeddings" require $\Omega(\log n)$ distortion
- Many others...

Useful for algorithms

A φ-expander decomposition of $G=(V,E)$ is $F \subseteq E$ s.t. that the Connected components of $(V, E \setminus F)$ are φ-expanders



Fact: $\forall$ m-edge graphs $\forall \varphi \in (0,1)$, $\exists$ a φ-expander decomposition F s.t. $|F| \leq O(\varphi \cdot \log_n \cdot m)$

Fact: Poly-time α-approximate sparsest cut $\rightarrow$ Poly-time φ-expander decomposition F s.t. $|F| \leq O(\alpha \cdot \varphi \cdot \log_n \cdot m)$