Today

- ℓ_1 Metrics + Bourgain for ℓ_1
- ℓ_1 vs ℓ_2
- Metrics as vectors + Convex cones
- $\ell_1(V) = \text{Cone}(\text{cut}(V))$

Recall

A metric space consists of a set of "points" V and function $f: V \times V \to \mathbb{R}_{\geq 0}$ satisfying

"metric" for short

1) $f(x,x) = 0 \quad \forall x \in V$ → really $f(x,y)=0$ iff $x=y$ (o/w a "pseudo metric")

2) $f(x,y) = f(y,x) \quad \forall x,y \in V$ (symmetric)

3) $f(x,y) \leq f(x,z) + f(z,y) \quad \forall x,y,z \in V$ (triangle inequality)

An embedding of metric space (V,d) into metric space (V',d') is a function $f: V \to V'$

(V,d) embeds isometrically into (V',d') iff $\exists$ embedding f s.t. $d(u,v) = d'(f(u), f(v)) \quad \forall u,v \in V$

Bourgain: Let $c \geq 1$ be a large constant chosen later

For $j \in [\log n]$ ($|B|$ possibilities)

For $i \in [c \cdot \log n]$ (repetitions)

S_{ij} contains each $v \in V$ independently w/ $\Pr \frac{1}{2^j}$

$\tilde{X}_{ij} = \delta(x, S_{ij})$ so $\tilde{x} \in \mathbb{R}^{c \cdot \log^2 n}$

Bourgain's Theorem: given any n-point metric (V,δ), $\exists$ (poly-time computable) embedding f w/ distortion $O(\log n)$ of (V,δ) into $(\tilde{V}, d)$ for $\tilde{V} \subseteq \mathbb{R}^{O(\log^2 n)}$

Euc. distance

Can reduce to $O(\log n)$ w/ JL

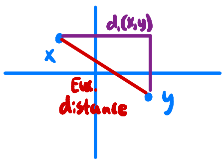
Expansion Claim: $\sum_{i,j} |\tilde{x}_{ij} - \tilde{y}_{ij}| \geq \frac{c}{40} \cdot \log n \cdot \delta(x,y) \quad \forall x,y$ except w/ $\Pr \leq \frac{1}{n}$

So far, have worked w/ "straight line" distances in $\mathbb{R}^n$

Today: ℓ_1 distances in $\mathbb{R}^n$ and their connection to cuts

ℓ_1 Metrics

Given $x, y \in \mathbb{R}^n$ define $d_1(x,y) := \sum_i |x_i - y_i|$



Claim: (V, d_1) is a metric space $\forall\, V \subseteq \mathbb{R}^n$

1) $d_1(x,x) = \sum_i |x_i - x_i| = 0$

2) $d_1(x,y) = \sum_i |x_i - y_i| = \sum_i |y_i - x_i| = d_1(x,y)$

3) $d_1(x,y) = \sum_i |x_i - y_i| = \sum_i |x_i - z_i + z_i - y_i| \underset{|a+b| \le |a| + |b|\ \forall a,b}{\le} \sum_i |x_i - z_i| + |z_i - y_i| = d_1(x,z) + d_1(z,y)$

Skippable

Bourgain's ℓ_1 Theorem: given any n-point metric (V, δ), $\exists$ (poly-time computable) embedding f w/ distortion $O(\log n)$ of (V, δ) into $(\tilde{V}, d_1)$ for $\tilde{V} \subseteq \mathbb{R}^{O(\log^2 n)}$

Let $\hat{x} := \frac{\tilde{x}}{\log n}$ where $\tilde{x}$ from ℓ_2 embedding

WTS $\delta(x,y) \le d_1(\hat{x}, \hat{y}) \le O(\log n) \cdot \delta(x,y)$

So $d_1(\hat{x}, \hat{y}) = \sum_{i,j} |\hat{x}_{ij} - \hat{y}_{ij}| = \frac{1}{\log n} \sum_{i,j} \left| (\delta(x, S_{ij}) - \delta(y, S_{ij})) \right| \le \frac{1}{\log n} \sum_{i,j} \delta(x,y) = c \cdot \log n \cdot \delta(x,y)$

OTOH $d_1(\hat{x}, \hat{y}) = \frac{1}{\log n} \cdot d_1(\tilde{x}, \tilde{y}) \underset{\text{expansion claim}}{\ge} \frac{1}{\log n} \frac{c}{40} \cdot \log n \cdot \delta(x,y) \underset{c \text{ large}}{\ge} \delta(x,y)$

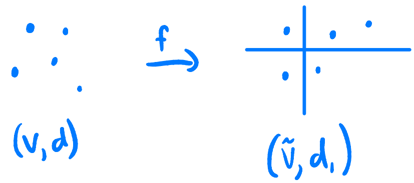
ℓ_1 vs ℓ_2

Euc. distance

$(\tilde{V}, d)$

ℓ_2 metric

(V,δ) is an ℓ_1 metric if it embeds isometrically into $(\tilde{V}, d_1)$ for $\tilde{V} \subseteq \mathbb{R}^n$ for some n



$\ell_2(V) \subseteq \mathbb{R}^{\binom{|V|}{2}}$

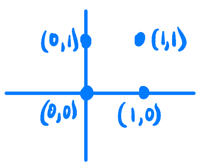
For arbitrary V, let $\ell_1(V) \subseteq \mathbb{R}^{\binom{|V|}{2}}$ be all ℓ_1 metrics of V

Fact: $\forall$ V have $\ell_2(V) \subseteq \ell_1(V)$

Rough sketch: random projections

Fact: $\exists$ V s.t. $\ell_1(V) \subsetneq \ell_2(V)$ → ℓ_1 has more "representational power" than ℓ_2

Sketch:



So what can we represent? Short answer: cuts

Metrics as Vectors

Claim: If (V,d) is a metric then $(V,\alpha\cdot d)$ is a metric $\forall \alpha>0$

1) $d(x,x)=0 \ \forall x\in V \rightarrow \alpha\cdot d(x,x)=0 \ \forall x\in V$
2) $d(x,y)=d(y,x) \ \forall x,y \rightarrow \alpha\cdot d(x,y)=\alpha\cdot d(y,x) \ \forall x,y$
3) $\alpha\cdot d(x,y)\underset{\uparrow}{\leq}\alpha(d(x,z)+d(z,y))=\alpha\cdot d(x,z)+\alpha\cdot d(z,y) \quad \forall x,y,z$

d triangle inequality

Skippable

Claim: If (V,d_1) and (V,d_2) are metrics then so is (V,d_1+d_2)

1) $d_1(x,x)+d_2(x,x)=0 \ \forall x$
2) $d_1(x,y)+d_2(x,y)=d_1(y,x)+d_2(y,x) \quad \forall x,y$
3) $d_1(x,y)+d_2(x,y)\underset{\uparrow}{\leq} d_1(x,z)+d_1(z,y)+d_2(x,z)+d_2(z,y) \quad \forall x,y,z$

d_1,d_2 triangle inequality

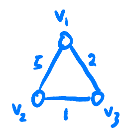
Skippable

Life Lesson: if can add + scale, good to interpret as vectors

1-1 Correspondance Between Metrics and Vectors

$$(V,d) \text{ s.t. } V=\{v_1,v_2,\dots,v_n\} \quad \leftrightarrow \quad \begin{pmatrix} d(v_1,v_2)\\ d(v_1,v_3)\\ \vdots \\ d(v_1,v_n)\\ d(v_2,v_3)\\ \vdots \end{pmatrix}\in\mathbb{R}^{\binom{n}{2}}$$

E.g.

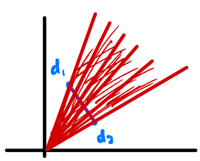


(V,d_G) $\leftrightarrow$ $\begin{pmatrix}5\\2\\1\end{pmatrix}$

Notation abuse: use $d\in\mathbb{R}^{\binom{n}{2}}$ for vector of (V,d) and call d a metric on V

Convex Cones

The <u>convex cone</u> of $D\subseteq\mathbb{R}^{\binom{n}{2}}$ is $\text{Cone}(D):=\left\{\sum_{d\in D}\alpha_d\cdot d : \alpha_d\geq 0 \ \forall d\right\}$



$D=\{d_1,d_2\}$

$[d_1,d_2]\in\text{Cone}(D)$

$X\in\text{Cone}(D)\rightarrow\alpha\cdot X\in\text{Cone}(D) \ \forall\alpha\geq 0$

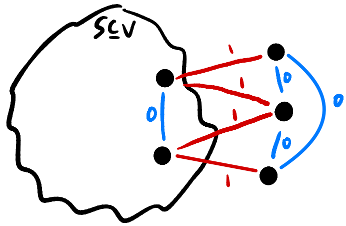
Corrolary: If D is a set of metrics on V, then $\tilde{d}\in\text{Cone}(D)$ is a $\overset{\text{really Pseudo metric...}}{\text{metric}}$ on V

Observe: If $d_1,d_2\in\text{Cone}(D)$ then $d_1+d_2\in\text{Cone}(D)$

$$d_1+d_2=\sum_{d\in D}\alpha_d d+\beta_d\cdot d=\sum_{d\in D}\underbrace{(\alpha_d+\beta_d)}_{\geq 0}\cdot d$$

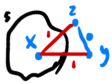
Cut+Line Metrics

(V,d) is a **cut metric** if $\exists\, S\subseteq V$ s.t.



$$d(u,v)=\begin{cases}0 & u,v\in S \text{ or } u,v\notin S\\ 1 & \text{otherwise}\end{cases}$$

A metric b/c (1), (2) trivial

3)  or 

$1\le 1$ $0 \le$ any Σ of distances

(V,d) is a **line metric** if it embeds isometrically into $(\hat V,\tilde d)$ for $\hat V\subseteq\mathbb{R}$ and $\tilde d(x,y)=|x-y|$ $\forall x,y\in\hat V$

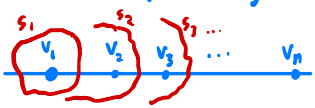
$\not\subseteq \mathbb{R}^k$

For arbitrary V, let $Cut(V)\subseteq\mathbb{R}^{\binom{|V|}{2}}$ be all cut metrics of V

Claim: If d is a line metric on V then $d\in Cone(cut(V))$

Notation abuse: use $v\in V$ and position on line interchangeably

Let $V=\{v_1,v_2,\dots,v_n\}$ be V sorted left $\to$ right and let $B:=v_n-v_1$

d: 

Let $S_i:=\{v_1,v_2,\dots,v_i\}$ and let d_i be corresponding cut metric for $i<n$

Let $\alpha_i:=v_{i+1}-v_i$ ≥ 0 by definition

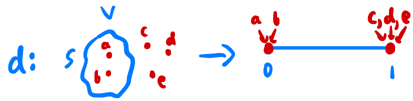
Have $d=\sum_i \alpha_i\cdot d_i$ b/c $\forall y\ge x$ $\sum_i \alpha_i d_i(v_x,v_y)=\sum_{i:\, S_i \text{ separates } v_x,v_y}(v_{i+1}-v_i)=\sum_{i\in[x,y)} v_{i+1}-v_i \underset{\text{telescoping}}{=} v_y-v_x=d(v_x,v_y)$

So $d\in Cone(cut(V))$

Theorem: $\ell_1(V) = \text{Cone}(\text{Cut}(V)) \;\; \forall V$ — Follows that ℓ_1 + line metrics are metrics by Corollary

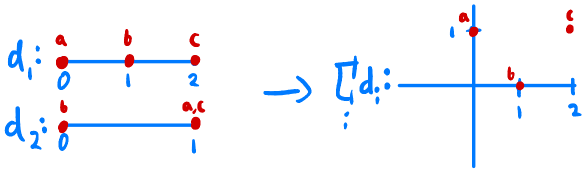
$\text{Cone}(\text{Cut}(V)) \subseteq \ell_1(V)$

Observe if d is a cut metric, it is also a line metric



Also if d is a line metric then so is $\alpha \cdot d \quad \forall \alpha \geq 0$

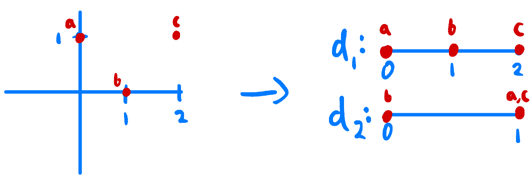
Also $\sum_i d_i$ for d_i a line metric is an ℓ_1 metric



So $d \in \text{Cone}(\text{Cut}(V)) = \underbrace{\sum_i \underbrace{\alpha \cdot \underbrace{d_i}_{\text{cut metric} \,=\, \text{line metric}}}_{\text{line metric}}}_{\ell_1 \text{ metric}}$ so $d \in \ell_1(V)$

$\ell_1(V) \subseteq \text{Cone}(\text{Cut}(V))$

If d is an ℓ_1 metric then $d = \sum_i d_i$ where d_i is a line metric



But each $d_i \in \text{Cone}(\text{Cut}(V))$ by Claim so by observation $\sum_i d_i = d \in \text{Cone}(\text{Cut}(V))$