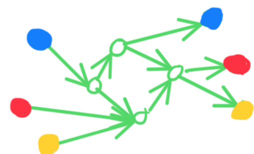
Fast Multicommodity Flow

- a primal-dual alg with MWU

MC Flow



(can generalize to non-unit integer capacities)

given directed graph $G=(V,E)$ with unit edge capacities (i.e. $\mu(e)=1 \; \forall e$) and pairs (s_i, t_i) where $s_i, t_i \in V$ $\forall i \in [k]$ for $k \geq 1$.

find a feasible flow for every (s_i, t_i) pair that maximizes the total flow

Recall "single commodity" max flow LP

$$\max \sum x_p \quad \text{s.t.}$$

$$\sum_{p \ni e} x_p \leq 1 \qquad \forall e \in E \quad \text{(unit capacities)}$$

$$x_p \geq 0 \qquad \forall p \in \mathcal{P}(s,t)$$

all s,t paths

$\boxed{\text{The dual was min-cut}}$

MC Flow LP

Let $\mathcal{P}_i$ be set of all s_i, t_i paths and $\mathcal{P} = \bigcup_{i \in [k]} \mathcal{P}_i$.

Primal

$$\max \sum_{i=1}^{k} \sum_{p \in \mathcal{P}_i} f_p$$

s.t.

$$\sum_{i=1}^{k} \sum_{\substack{p \in \mathcal{P}_i, \\ e \in p}} f_p \le 1 \quad \forall e \in E$$

$$f_p \ge 0 \quad \forall p \in \mathcal{P}$$

What are ℓ_e's?

Dual

$$\min \sum_{e \in E} \ell(e)$$

s.t.

$$\sum_{e \in p} \ell_e \ge 1 \quad \forall p \in \mathcal{P}$$

$$\ell_e \ge 0 \quad \forall e \in E$$

This is multicut.

(Can solve dual w/ ellipsoid)

(what sep oracle?)

PD alg

We give a fast approximation scheme

Defn: an approximation scheme for a maximization / minimization problem is an alg that takes $\varepsilon > 0$ as an input and returns a solution with value at least / most $(1-\varepsilon)$ / $(1+\varepsilon)$ OPT for any $\varepsilon > 0$.

Alg (GK 07)

Input : directed $G=(V,E)$, $|E|=m$, $\mu: E \to \mathbb{R}$, $\{(s_i,t_i)\}_{i\in[k]}$, $\varepsilon > 0$

Initialize $f=0$ and $\ell_e = \delta := (1+\varepsilon)^{1-\frac{1}{\varepsilon}} m^{-\frac{1}{\varepsilon}}$ $\forall e \in E$

while ℓ is not feasible for Dual:

ⓐ let p be the shortest path in $\mathcal{P}$ under ℓ

$\left(\text{i.e., } p := \arg\min_{p \in \mathcal{P}} \sum_{e \in p} \ell(e)\right)$

ⓑ update f to send 1 flow thru p

$(\text{i.e., } f_p = 1)$

ⓒ MWU each $e \in p$

$(\text{i.e., } \ell_e \leftarrow (1+\varepsilon)\, \ell_e \quad \forall e \in p)$

endwhile

return $f \cdot \dfrac{1}{\log_{1+\varepsilon}\left(\frac{1+\varepsilon}{\delta}\right)}$ (scaling the flow)

let $\tilde{f}$ be this scaled flow

f : "unscaled flow"

We need to show that this alg

① terminates

② feasible

③ $1-\varepsilon$ apx

Termination Lemma : Alg terminates after at most $m \log_{1+\varepsilon}\left(\frac{1+\varepsilon}{\delta}\right)$ iterations.

Pf: alg termin8s when $l_e \geq 1 \ \forall e$. fix some e.

- l_e starts at δ and ends at most $1+\varepsilon$

  because l_e is only increased if $e \in$ a shortest path p where $\sum_{e \in p} l_e < 1$ and increased by at most $* \, 1+\varepsilon$

$\Rightarrow$ # of times we can send 1 flow down an edge e is $\leq$ # times δ can be multiplied before it hits $1+\varepsilon$

$= \log_{1+\varepsilon}\left(\frac{1+\varepsilon}{\delta}\right)$ times

- Since we send 1 flow down an edge each iteration

  $\Rightarrow$ # iters $\leq$ # times we can send 1 flow $\forall$ edges

$\Rightarrow$ there are m edges, so all edges have $l_e \geq 1+\varepsilon$ after $m \log_{1+\varepsilon}\left(\frac{1+\varepsilon}{\delta}\right)$ iters ∎

Feasibility Lemma: Alg $\left(\text{i.e. } \tilde{f} := \frac{f}{\log_{1+\varepsilon}\left(\frac{1+\varepsilon}{\delta}\right)}\right)$ is feasible.

Pf: fix some e. WTS $\sum_{i\in[k]} \sum_{\substack{p\in\mathcal{P}_i,\\ e\in p}} \tilde{f}_p \le 1$

(just e's primal constraint)

afsoc LHS > 1

$\Rightarrow \sum \sum f_p > \log_{1+\varepsilon}\left(\frac{1+\varepsilon}{\delta}\right)$

$\sum\sum f_p$ ← # of paths that we send 1 flow down

we e init

but $\ell_e = \delta(1+\varepsilon)^{\sum\sum f_p}$

$> \delta(1+\varepsilon)^{\log_{1+\varepsilon}\left(\frac{1+\varepsilon}{\delta}\right)} = \delta\left(\frac{1+\varepsilon}{\delta}\right)$

$= 1+\varepsilon$

$\ell_e > 1+\varepsilon$ contradicts while loop condition

∎

Approximation lemma : Alg (aka $\tilde{f}$) is $(1-\varepsilon)$-apx.

aka length of shortest path ≤ 1

Pf: • let $\lambda_i := \min_{p \in P} \sum_{e \in p} \ell(e)$ at beginning of i^{th} iter

• let $D_i := \sum_{e \in E} \ell_e$ aka current dual value

• let OPT be opt val of dual (and primal, bc of ...)

• If we scale $\frac{\ell_e}{\lambda_i}$ $\forall e$, we get a dual feasible solution. Because λ_i is min path length so all paths get total length ≥ 1 from scaling

$\Longrightarrow \text{OPT} \leq \frac{D_i}{\lambda_i}$ since $\frac{D_i}{\lambda_i}$ is the value of feasible scaled solution

(*) $\Longrightarrow \lambda_i \leq \frac{D_i}{\text{OPT}}$

• But $D_{i+1} \overset{c}{=} (1+\varepsilon) \sum_{e \in p} \ell_e + \sum_{g \notin p} \ell_g$

shortest path in this round

$$= D_i + \varepsilon \left(\sum_{e \in p} \ell_e \right)$$ by defn of D_i

$$= D_i + \varepsilon \lambda_i$$ by defn of λ_i

$$\overset{(*)}{\leq} D_i + \varepsilon \frac{D_i}{\text{OPT}}$$

(*) $$= D_i \left(1 + \frac{\varepsilon}{\text{OPT}} \right)$$ just factor out let T be the last iteration

(dual feasible) (*) and initially $D_{\rightharpoonup} = m\delta$

$$\Rightarrow 1 \le D_T \le m\,\delta\left(1+\frac{\varepsilon}{OPT}\right)^{T}$$

$$\Rightarrow 1 \le m\,\delta\left(1+\frac{\varepsilon}{OPT}\right)^{T}$$

take logs

$$\Longrightarrow 0 \le \ln m + \ln\delta + T\ln\left(1+\frac{\varepsilon}{OPT}\right)$$

$$\approx \ln m + \ln\delta + \frac{\varepsilon T}{OPT}$$

$e^x \approx 1+x$ for small x $\Leftrightarrow x \approx \ln 1+x$

$$\Longrightarrow \frac{OPT}{\varepsilon}\left(\ln\frac{1}{\delta} - \ln m\right) \le T$$

rearranging

log rule

$$\Rightarrow \frac{OPT}{\varepsilon}\cdot\ln\left(\frac{1}{\delta m}\right) \le T$$

turn the $\frac{1}{\varepsilon}$ into $\log_{1+\varepsilon}$ cuz $\frac{\ln x}{\ln 1+\varepsilon} \rightarrow \approx \ln e^{\varepsilon} \approx \varepsilon$

$$\Longrightarrow OPT\cdot\log_{1+\varepsilon}\left(\frac{1}{\delta m}\right) \le T$$

Finally use $\delta = (1+\varepsilon)^{1-\frac{1}{\varepsilon}} m^{-\frac{1}{\varepsilon}}$

$$\Rightarrow OPT\cdot\log_{1+\varepsilon}\left[\frac{1}{(1+\varepsilon)^{1-\frac{1}{\varepsilon}}\, m^{-\frac{1}{\varepsilon}}\, m}\right] \le T$$

log rules and factoring around

$$\Rightarrow OPT\cdot\left(\frac{1}{\varepsilon}-1\right)\cdot\log_{1+\varepsilon}\left(m(1+\varepsilon)\right) \le T$$

Observe now that $T \le f$ upon termination bc at each iter we add 1 to total flow

$$\Rightarrow \underbrace{OPT \cdot \left(\frac{1}{\varepsilon} - 1\right) \log_{1+\varepsilon}\left(m(1+\varepsilon)\right)}_{\text{approximation factor of unscaled flow } f} \le f$$

Now compare LHS to $\log_{1+\varepsilon}\left(\frac{1+\varepsilon}{\delta}\right)$

We have $\log_{1+\varepsilon}\left(\frac{1+\varepsilon}{\delta}\right) = \log_{1+\varepsilon}\left(\frac{1+\varepsilon}{m^{-\frac{1}{\varepsilon}}(1+\varepsilon)^{1-\frac{1}{\varepsilon}}}\right)$

$$= \frac{1}{\varepsilon} \cdot \log_{1+\varepsilon}\left(m(1+\varepsilon)\right)$$

Scale factor

Compute apx ratio:

$$\frac{OPT\left(\frac{1}{\varepsilon} - 1\right) \log_{1+\varepsilon}\left(m(1+\varepsilon)\right)}{\frac{1}{\varepsilon} \log_{1+\varepsilon}\left(m(1+\varepsilon)\right)}$$

$$= OPT \cdot \frac{\frac{1}{\varepsilon} - 1}{\frac{1}{\varepsilon}} = OPT(1-\varepsilon)$$

approximation of scaled flow, aka $\tilde{f}$

∎

Main theorem (putting it all together)

There is an alg getting $(1-\varepsilon)$-apx for max MC flow in time

$$O\left(T \cdot k \cdot \frac{1}{\varepsilon^2} \cdot m \log m\right)$$

where T is time to compute shortest s-t path

pf: # of rounds $\leq m \log_{1+\varepsilon}\left(\frac{1+\varepsilon}{\delta}\right)$

$$= m \cdot \frac{1}{\varepsilon} \cdot \log_{1+\varepsilon}\left(m(1+\varepsilon)\right)$$

$$\approx \frac{\log m}{\varepsilon}$$

In each round must compute a shortest path $\forall$ k st pairs in time $T \cdot k$