Today

1) The Set Cover Problem
2) Solving (1) using LP framework + randomization
3) Solve (1) using Primal-dual

Recall

<u>Claim:</u> If $K = \left\{x : \begin{array}{l} Ax \le b \\ x \ge 0 \end{array}\right\}$ then K has a BFS

Further, give c w/ $OPT := \max_{y \in K} \langle c, y \rangle \ne \infty$

then $\exists$ a BFS z s.t. $\langle c, z \rangle = OPT$

Efficiently Computable

The LP Framework

1) Express Problem as an IP
2) Get a $_{\wedge}^{\text{Optimal}}$ BFS solution x to LP relaxation of (1) (Must assume 1 exists)
3) Use x to solve Problem (Sometimes only approximately)

The Concentration Framework

To show fact $(*)$

a) Show $(*)$ true if all RVs near $\mathbb{E}$

b) Concentration: each RV at $\mathbb{E}$ $(\pm \log n)$ w/ good Probability

c) Union bound: all RVs " "

Chernoff Bound

Let $X_1, X_2, \dots, X_n$ be independent RVs s.t. $X_i = \begin{cases} 1 & \text{w/ Pr } p_i \\ 0 & \text{o/w} \end{cases}$

Let $X := \sum_i X_i$, $\mu := \mathbb{E}[X]$

And $\forall\ \delta \in (0,1)$ $\Pr(X \le (1-\delta)\cdot\mu) \le \exp(-\delta^2 \mu / 2)$

Dual LP

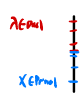
$\overset{\min}{\cancel{\max}}$ $\overset{\langle -c, x \rangle}{\langle c, x \rangle}$ s.t. $Ax \le b$

(Primal)

$\overset{\max}{\cancel{\min}}$ $\overset{\langle -b, \lambda \rangle}{\langle b, \lambda \rangle}$ s.t. $A^T\lambda = c^T$, $\lambda \ge 0$

(Dual)

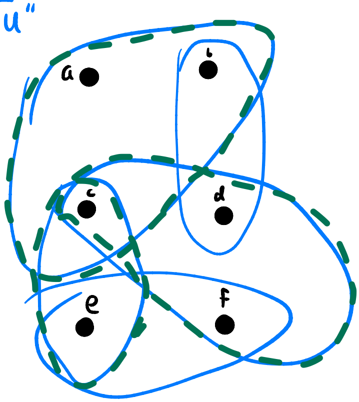
<u>Weak Duality:</u> Always have $\overset{D' \le P'}{P \le D}$ if $\overset{P', D' = -P, -D}{P, D}$ are Primal, dual Optimal values



The Set Cover Problem

Given universe U, sets $B \subseteq 2^U$, set weights $w: B \to \mathbb{R}_{\geq 0}$ $\quad m := |B|$

Want: $A \subseteq B$ s.t. $\underbrace{\bigcup_{S \in A} S = U}_{\text{"A covers U"}}$ minimizing $w(A) := \sum_{S \in A} w(S)$ $\quad n := |U|$



$U \quad B \quad A$

Let $\mathrm{OPT} := \min_{A : \bigcup_{S \in A} S = U} w(A)$

Problem: Set cover is NP-hard

Still possible: Find $A \subseteq B$ covering U s.t. $w(A) \leq \alpha \cdot \mathrm{OPT}$ for small $\alpha > 1$ in P

e.g. all set cover instances

↳ Given a family of minimization problems Π w/ objective w, a randomized algorithm $\mathcal{A} : \Pi \to \text{Solutions}$ is an <u>α-approximation</u> if $\mathbb{E}[w(\mathcal{A}(P))] \leq \alpha \cdot \mathrm{OPT}(P)$ $\forall P \in \Pi$

Today: Use LP-framework to do this

<u>Theorem 1:</u> $\exists$ a randomized $\mathrm{Poly}(n,m)$-time <u>$O(\log n)$</u>-approximation for set cover

↳ must be $\Omega(\log n)$ assuming $P \neq NP$

<u>Theorem 2:</u> $\exists$ a $\mathrm{Poly}(n,m)$-time f-approximation for set cover

where $f = \max_{v \in U} \left|\{S \in B : v \in S\}\right|$

Set Cover IP/LP

Variable X_S $\forall S \in B$ so $x \in \mathbb{R}^m$, let $w(x) := \sum_{S \in B} w(S) \cdot x_S$, $OPT_{LP} := \max_{x \in K_{SC}} \langle w, x \rangle$

so $OPT_{LP} \leq OPT$

IP $\min w(x)$ s.t.

$$\sum_{S: v \in S} x_S \geq 1 \quad \forall v \in U$$
$$X_S \in \{0,1\} \quad \forall S \in B$$

$\implies$

LP $\min w(x)$ s.t.

$$\left.\begin{array}{ll} \sum_{S: v \in S} x_S \geq 1 & \forall v \in U \\ \cancel{X_S \in [0,1]} & \forall S \in B \\ X_S \geq 0 & \end{array}\right] := K_{SC}$$

Randomized Rounding:

$\to$ (3) today

$$x \in K_{SC} \xrightarrow{\text{randomization}} \tilde{x} \in K_{SC} \cap \mathbb{Z}^n$$

Claim: given $x \in K_{SC}$, can poly(m,n) time compute a random $\tilde{x}$ s.t.

$\tilde{x} \in K_{SC} \cap \mathbb{Z}^n$ w/ $\Pr \geq 1 - \frac{1}{n}$ and $\mathbb{E}[w(\tilde{x})] \leq O(\log n) \cdot w(x)$

Proof of Thm 1 w/ Claim

(1) Consider Set Cover IP

b/c $w(x) \leq$ IP optimal

(2) Compute optimal $x \in K_{SC}$ so $w(x) \leq OPT$

(3) Then $\tilde{w}$ corresponds to a solution $A \subseteq B$ for set cover s.t.

$$w(A) = w(\tilde{x}) \leq O(\log n) \cdot w(x) = O(\log n) \cdot OPT_{LP} \leq O(\log n) \cdot OPT$$

Proof of Claim

Let $\tilde{x}_S := \begin{cases} 1 & \text{w/ Pr } \min(10 \cdot \ln n \cdot x_S, 1) \\ 0 & \text{o/w} \end{cases}$

$\tilde{x}$ trivially poly(n,m) time computable

$$\mathbb{E}[w(\tilde{x})] = \sum_{S \in B} \mathbb{E}[w(\tilde{x}_S)] = 10 \cdot \ln n \cdot \sum_{S \in B} x_S \cdot w(S) = O(\log n) \cdot w(x)$$

$\tilde{x} \in K_{SC} \cap \mathbb{Z}^n$ w/ $\Pr \geq 1 - \frac{1}{n}$ $\to$ Note $\tilde{x} \in \mathbb{Z}^n$ and $\tilde{x}_S \in \{0,1\}$ $\forall S \in B$ trivially

(a) Fix $v \in U$, let $\tilde{x}_v := \sum_{S \in B: v \in S} \tilde{x}_S$ so $\mathbb{E}[\tilde{x}_v] \geq 16 \cdot \ln n$

(b) By Chernoff, $\Pr(\tilde{x}_v \leq 8 \cdot \ln n) \leq \Pr(\tilde{x}_v \leq \frac{1}{2} \cdot \mathbb{E}[\tilde{x}_v]) \leq \exp(-16 \cdot \ln n / 8) = \frac{1}{n^2}$

(c) By Union bound $\tilde{x}_v \geq 1$ $\forall v \in U$ except w/ $\Pr \leq \frac{1}{n}$ so $\tilde{x}_v \geq 1$ $\forall v \in U$ except w/ $\Pr \leq \frac{1}{n}$

Set Cover Dual

Variable X_S $\forall S \in B$ so $x \in \mathbb{R}^m$

LP Min $w(x)$ s.t.

$$\sum_{S: v \in S} x_S \geq 1 \quad \forall v \in U$$

$$X_S \geq 0 \quad \forall S \in B$$

A "Covering" LP

$\longleftrightarrow$

Variable y_v for each $v \in U$ so $y \in \mathbb{R}^n$

Dual Max $\sum_{v \in U} y_v$ s.t.

$$\sum_{v \in S} y_v \leq w(v) \quad \forall S \in B$$

$$y_v \geq 0 \quad \forall v \in U$$

A "Packing" LP

→ See later for a derivation of dual

Given $A \in \mathbb{R}_{\geq 0}^{m \times n}$, $b \in \mathbb{R}_{\geq 0}^m$, $c \in \mathbb{R}_{\geq 0}^n$

min $\langle c, x \rangle$ s.t. $Ax \geq b, x \geq 0$ is a <u>Covering LP</u>

max $\langle c, y \rangle$ s.t. $Ay \leq b, y \geq 0$ is a <u>Packing LP</u>

duals of eachother (w/ different A, b, c)

Primal-Dual Approach:

grow feasible dual soln. + use dual soln. to construct similar cost primal

↳ Note: don't always have to solve LP

<u>Claim:</u> Can Poly(n, m) time compute a set cover solution $A \subseteq B$ and a feasible dual solution y s.t. $w(A) \leq f \cdot \sum_{v \in U} y_v$

Proof of thm. 2 w/ Claim

A from claim is Poly(n, m) computable

$D \leq P$ by weak duality

Have $w(A) \leq f \cdot \sum_{v \in U} y_v \leq f \cdot OPT_{LP} \leq f \cdot OPT$ so an f-approximation

Proof of Claim

Given dual soln. y, say $S \in \mathcal{B}$ is **tight** if $\sum_{v \in S} y_v = w(S)$

P-D Alg.

$y \leftarrow 0$

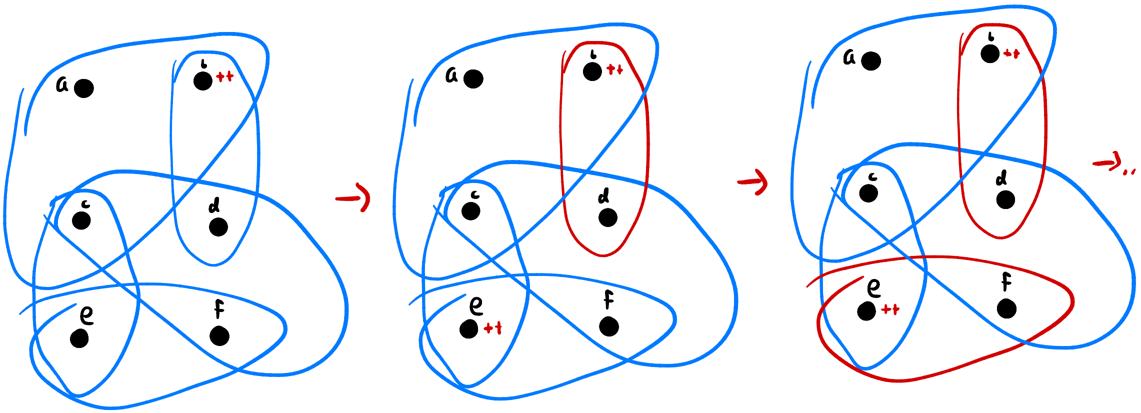
$A \leftarrow \emptyset$

While $\exists\, v \in U$ not covered by A

  Increase y_v until $\exists\, S \in \mathcal{B}$ covering v that is tight → Specifically, let $y_v \leftarrow y_v + \epsilon$ where $\epsilon := \min_{S: v \in S}\left(w(S) - \sum_{v \in S} y_v\right)$

  Add this S to $\mathcal{B}$

Return A, y



Notice at start of iteration v not coverd by A but v is covered at end

→ Alg is $\text{poly}(n,m)$ time

→ Also A is a set cover

y is a feasible dual solution by construction

$$w(A) \le f \cdot \sum_{v \in U} y_v$$

If $S \in A$, then $\sum_{v \in S} y_v = w(S)$ by construction

So $w(A) = \sum_{S \in A} w(S) \overset{\text{above line}}{=} \sum_{S \in A} \sum_{v \in S} y_v \overset{\text{switch } \Sigma \text{ order}}{=} \sum_{v \in U} \sum_{S: v \in S} y_v = \sum_{v \in U} y_v \sum_{S: v \in S} 1 \overset{f \text{ defn.}}{\le} f \cdot \sum_{v \in U} y_v$

Deriving dual of SC LP

Primal

$$\min \ w(x) \ \text{ s.t.} \qquad\Rightarrow\qquad \max \ \langle -w, x\rangle \ \text{ s.t.}$$

$$\sum_{S: v \in S} x_S \geq 1 \quad \forall v \in U \qquad\Rightarrow\qquad -\sum_{S: v \in S} x_S \leq -1 \quad \forall v \in U \quad (y_v)$$

$$X_S \geq 0 \quad \forall S \in B \qquad\qquad -X_S \leq 0 \quad \forall S \in B \quad (\lambda_S)$$

Dual

$$\min \ -\sum_{v \in S} y_v \ \text{ s.t.} \qquad\qquad \max \ \sum_{v \in S} y_v \ \text{ s.t.}$$

$$-\sum_{v \in S} y_v - \lambda_S = -w(S) \quad \forall S \in B \qquad\Rightarrow\qquad \sum_{v \in S} y_v \leq w(S) \quad \forall S \in B$$

$$y, \lambda \geq 0 \qquad\qquad y \geq 0$$