Today

1) Moving Around in LPs

2) Basic Feasible Solutions (BFSs)

3) Feasibility in exp. time via BFSs

4) Optimality of a BFS / optimization in exp. time

5) Rank Lemma

Recall

LP feasibility: decide if $K = \{x : \underset{m \times n}{A}x \le b\} = \emptyset$

LP optimization: return $x \in K$ s.t. $\langle c, x \rangle = \mathrm{OPT}$ (or report $\mathrm{OPT} = \infty$)

Claim 1: Given $x \in K = \{x : Ax \le b\}$ and $w \in \mathbb{R}^n$ s.t.

$$\langle w, a_i \rangle \le 0 \quad \forall a_i \in rows(A)$$

Have $x + \varepsilon \cdot w \in K \quad \forall\ \varepsilon \ge 0$

For $a_i \in rows(A)$, have $\langle a_i, x + \varepsilon w \rangle = \langle a_i, x \rangle + \varepsilon \cdot \langle a_i, w \rangle$

$$\le \langle a_i, x \rangle$$

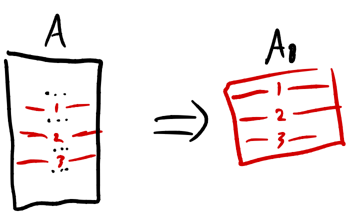
$$\le b_i$$

Tight Constraints

$x \in K$ is tight for constraint $\langle a_i, x \rangle \le b$ iff $\langle a_i, x \rangle = b$

$Tight(x) := \{a_i \in rows(A) : \langle a_i, x \rangle = b_i\}$

For $B \subseteq rows(A)$ let A_B be matrix w/ rows B



$rows(A_B) = B$

Will consider $A_{Tight(x)}$

Claim 2: For $x \in K$, if $T = Tight(x)$ and $w \in Ker(A_T)$ then $Tight(x) \subseteq Tight(x + \varepsilon \cdot w)$ $\forall \varepsilon \ge 0$

$w \in Ker(A_T) \to A_T w = 0 \to \langle a_i, w \rangle = 0 \quad \forall\ a_i \in T$

So $\forall\ a_i \in Tight(x)$ have $\langle a_i, x + \varepsilon w \rangle = \langle a_i, x \rangle + \varepsilon \langle a_i, w \rangle = \langle a_i, x \rangle = b_i$

So $a_i \in Tight(x + \varepsilon w)$

Lem 1: Given $x \in K = \{x : Ax \le b\}$ w/ $T := \text{Tight}(x)$ and $w \in \text{Ker}(A_T)$

If $\langle w, a_i \rangle \le 0 \quad \forall \quad a_i \in \text{rows}(A)$

Then $\forall\ \varepsilon \ge 0 \quad x + \varepsilon w \in K$ and $\text{Tight}(x) \subseteq \text{Tight}(x + \varepsilon w)$

Immediate from Claim 1+2

Lem 2: Given $x \in K = \{x : Ax \le b\}$ w/ $T := \text{Tight}(x)$ and $w \in \text{Ker}(A_T)$

If $\langle w, a_i \rangle > 0$ for some $a_i \in \text{rows}(A)$

Then $\exists \varepsilon > 0$ s.t. $x + \varepsilon' w \in K \quad \forall \ \varepsilon' \in [0, \varepsilon]$

and $\text{Tight}(x) \subsetneq \text{Tight}(x + \varepsilon w)$

Got to here in class

(but also did optimization → search → feasibility reduction)

Let $I_w := \{ i : \langle a_i, w \rangle > 0 \}$, let $\varepsilon_i := \frac{b_i - \langle x, a_i \rangle}{\langle w, a_i \rangle}$ for $i \in I_w$

Let $\varepsilon = \min_{i \in I_w} \varepsilon_i$ and $y = x + \varepsilon' w$ for $\varepsilon' \le \varepsilon$

$y \in K$

If $a_i \in T$ then $\langle y, a_i \rangle = b_i$ by Claim 2

If $a_i \notin T$ then
$$
\begin{aligned}
\langle y, a_i \rangle &= \langle x, a_i \rangle + \varepsilon' \langle w, a_i \rangle \\
&\le \langle x, a_i \rangle + \varepsilon \cdot \langle w, a_i \rangle \quad (1) \\
&\le \langle x, a_i \rangle + \varepsilon_i \cdot \langle w, a_i \rangle \quad (2) \\
&= b_i
\end{aligned}
$$

Also for $i \in I_w$ s.t. $\varepsilon_i = \varepsilon$ (of which ≥ 1) and $\varepsilon' = \varepsilon$, (1), (2) w/ equality so $\text{Tight}(x) \subsetneq \text{Tight}(x + \varepsilon w)$

Thm: Given $x \in K = \{x : Ax \le b\}$ w/ $T := \text{Tight}(x)$ and $w \in \text{Ker}(A_T)$

$\exists \delta > 0$ s.t. $\underbrace{x \pm \delta \cdot w}_{\{x+\delta w,\, x-\delta w\}} \subseteq K$

Further, if $\langle w, a_i \rangle \neq 0$ for some $a_i \in \text{rows}(A)$ then $\text{Tight}(x) \subsetneq \text{Tight}(y)$

for some $y \in x \pm \delta \cdot w$

Say w type 1 if $w \in \text{Ker}(A_T)$ and $\langle w, a_i \rangle \le 0 \ \forall \ a_i \in \text{rows}(A)$

type 2 if $w \in \text{Ker}(A_T)$ and $\langle w, a_i \rangle > 0$ for some $a_i \in \text{rows}(A)$

$w \in \text{Ker}(A) \to -w \in \text{Ker}(A)$ so each of $w, -w$ of type 1 or 2

Suppose both type 1

Lem 1 gives $x \pm \varepsilon w \subseteq K \ \forall \ \varepsilon \ge 0$

If $\langle w, a_i \rangle \neq 0$ for some $a_i \in \text{rows}(A)$ then

at least 1 of $w, -w$ of type 2, WLOG w

Suppose w type 2, $-w$ type 1

Lem 2 gives ε s.t. $x + \varepsilon w \in K$ and $\text{Tight}(x+\varepsilon w) \supsetneq \text{Tight}(x)$

Lem 1 gives $x - \varepsilon w \in K$

Suppose both type 2

Lem 2 gives ε_1 s.t. $x + \varepsilon_1' w \in K \ \forall \varepsilon_1' \in [0, \varepsilon_1]$ and $\text{Tight}(x+\varepsilon_1 w) \supsetneq \text{Tight}(x)$

Lem 2 gives ε_2 s.t. $x - \varepsilon_2' w \in K \ \forall \varepsilon_2' \in [0, \varepsilon_2]$ and $\text{Tight}(x-\varepsilon_2 w) \supsetneq \text{Tight}(x)$

Let $\delta = \min(\varepsilon_1, \varepsilon_2)$

Basic Feasible Solutions

$x \in K$ is a <u>basic feasible solution (BFS)</u> iff $\dim(\text{Tight}(x)) = n$

<u>Claim:</u> If $K = \left\{ \begin{matrix} Ax \le b \\ x \ge 0 \end{matrix} \right\} \neq \emptyset$ then K has a BFS

↳ Special Case: equational form LPs

Further, give c w/ $\text{OPT} := \max_{y \in K} \langle c, y \rangle \neq \infty$

then $\exists$ a BFS z s.t. $\langle c, z \rangle = \text{OPT}$

Notice if $A' = \begin{pmatrix} A \\ -I \end{pmatrix} = \begin{pmatrix} A \\ --e_1-- \\ --e_2-- \\ \vdots \\ --e_n-- \end{pmatrix}$, $b' = \begin{pmatrix} b \\ 0 \end{pmatrix}$ then $K = \{x : A'x \le b'\}$

Let $x \in K$ be any <u>optimal</u> solution maximizing $|\text{Tight}_{A'}(x)| := |T|$

↳ well defined b/c $\text{OPT} \neq \infty$

AFSOC x not a BFS so $\dim(T) < n$ so $\text{rank}(A_T) < n$

$\text{rank}(A_T) < n$ so $\exists w \neq 0$ s.t. $w \in \text{Ker}(A_T)$

But if $w_i \neq 0$ then $\langle -e_i, w_i \rangle \neq 0$ so $\exists a_j \in \text{rows}(A')$ s.t. $\langle a_j, w \rangle \neq 0$

so by theorem then $\exists y \in K$ w/ $\text{Tight}(x) \subsetneq \text{Tight}(y)$ → Contradicts choice of x

and $x \pm \epsilon w \subseteq K$

If $\langle c, w \rangle \neq 0$ then either $\langle c, w \rangle > 0$ or $\langle c, -w \rangle > 0$

$\rightarrow \langle x + \epsilon w, c \rangle > \text{OPT}$ or $\langle x - \epsilon w, c \rangle > \text{OPT}$ ← Both contradicting

So $\langle c, w \rangle = 0$

But then $\exists y \in K$, $\langle c, y \rangle = \text{OPT}$ and $\text{Tight}(x) \subsetneq \text{Tight}(y)$

↳ A contradiction to choice of x

Claim: If x is a BFS of $\{x : Ax \le b\}$ then it is the unique soln. to
$$A_B x = b_B$$
$\forall\, B \subseteq \text{Tight}(x)$ a basis of $\mathbb{R}^n$

$A_B x = b_B$ by definition of Tight(x)

Uniqueness from A_B a full rank $n \times n$ matrix

converse of above

Claim: If $x \in \mathbb{R}^n$ satisfies $A_B x = b_B$ for some basis $B \subseteq \text{rows}(A)$ of $\mathbb{R}^n$ and $x \in K$ then x is a BFS

B/c tight for $B \subseteq \text{rows}(A)$ so $\dim(\text{Tight}(x)) \ge \text{rank}(A_B) = n$

Enum. BFS Alg

$\mathcal{B} = \emptyset$

$\forall$ bases $B \subseteq \text{rows}(A)$ of $\mathbb{R}^n$

If $\exists x$ s.t. $A_B x = b_B$ and $x \in K$, $\mathcal{B} \leftarrow \mathcal{B} + x$

Return $\mathcal{B}$

Use Gaussian Elimination + note that if $\exists x$ s.t. $A_B x = b_B$ then ≤ 1 such x (+ check if $\in K$)

Correct b/c x a BFS iff $\exists$ basis $B \subseteq \text{rows}(A)$ of $\mathbb{R}^n$ s.t. $A_B x = b_B$ and $x \in K$

Runtime $= \binom{m}{n} \cdot \text{Poly}(n, m) \le m^n \cdot \text{Poly}(n, m)$

Feasibility / Optimization Alg.

Put LP into form $\left\{ x : \begin{matrix} Ax \le b \\ x \ge 0 \end{matrix} \right\}$

Note Putting into above form doubles n

Let $\mathcal{B}$ be all BFSs of

Return $K \ne \emptyset$ iff $\mathcal{B} \ne \emptyset$ / return $x \in \mathcal{B}$ maxing $\langle x, c \rangle$ (or report OPT $= \infty$)

Correct by fact that $\left\{ x : \begin{matrix} Ax \le b \\ x \ge 0 \end{matrix} \right\}$ feasible iff a BFS

Takes $\approx m^{2n} \cdot \text{Poly}(n, m)$ time (vs $\approx n^{2^n}$ time last time)

Got to here in class (only stated BFS optimality)

Rank Lemma

Given $K = \left\{ x : \begin{matrix} Ax \le b \\ \ell \le x \le r \end{matrix} \right\}$, if x is a BFS of K

then $|\{i : \ell_i < x_i < r_i\}| \le \text{rank}(A)$

Let $S := \{i : \ell_i < x_i < r_i\}$ and $\bar{S} := [n] \setminus S$

Let $A' = \begin{pmatrix} A \\ I \\ -I \end{pmatrix}$ and $b' = \begin{pmatrix} b \\ r \\ \ell \end{pmatrix}$ so $K = \{x : A'x \le b'\}$

Consider a basis $B \subseteq \text{Tight}(x) \rightarrow |B| = n$

$|B \cap \text{rows}(A)| \le \text{rank}(A)$ since B independent

Thus $\underbrace{|B \cap \text{rows}(I)| + |B \cap \text{rows}(I')|}_{=|\bar{S}|} \ge n - \text{rank}(A)$

So $|S| + |\bar{S}| = n \rightarrow |S| \le n - |\bar{S}| \le \text{rank}(A)$