Today

1) Max bipartite matching
2) Integer Programming + LP Integrality
3) The LP Framework
4) Solving (1) w/ (3)

Recall

Claim: If $K = \left\{ x : \begin{matrix} Ax \leq b \\ x \geq 0 \end{matrix} \right\}$ then K has a BFS

Further, give c w/ $OPT := \max_{y \in K} \langle c, y \rangle \neq \infty$

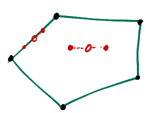
then $\exists$ a BFS z s.t. $\langle c, z \rangle = OPT$

$\hookrightarrow$ Efficiently Computable

Fact: $x \in K = \{x : Ax \leq b\}$ is a BFS iff it is a vertex

iff it is an extreme point

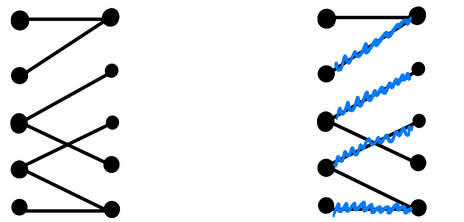
Given $K \subseteq \mathbb{R}^n$, x is an extreme point of K if $x \in [y, z]$ for $y, z \in K \rightarrow y = z$



Max Bipartite Matching

Given bipartite graph $G=(V,E)$, find a matching $M \subseteq E$ maximizing $|M|$

↳ No 2 distinct edges share an endpoint



G   a Matching M

An Equivalent Problem

Let $E=\{e_1, e_2, \ldots, e_m\}$

For $x \in \mathbb{R}^m$, notate x_i as x_{e_i}

Call this Problem IP_M

Find $x \in \mathbb{R}^m$ maximizing $\sum_{e \in E} x_e$ s.t.

$$x(\delta(u)) \le 1 \quad \forall u \in V \qquad \left(x(\delta(u)) := \sum_{e \in \delta(u)} x_e\right)$$

$$x_e \in \{0,1\} \quad \forall e \in E$$

1-1 correspondance between these x and matchings

$x \to$ Matching $\{e : x_e = 1\}$

$M \to x$ where $x_e = \begin{cases} 1 & \text{if } e \in M \\ 0 & \text{o/w} \end{cases}$

Not a linear program b/c $x_e \in \{0,1\}$ not a linear constraint

IP_M is an "integer" program

Integer Programming

Suppose given $K = \{x : Ax \leq b\}$ and $c \in \mathbb{R}^n$
$m \times n$

let $OPT_{LP}(c) := \max_{x \in K} \langle c, x \rangle$ and $OPT_{IP}(c) := \max_{x \in K \cap \mathbb{Z}^n} \langle c, x \rangle$

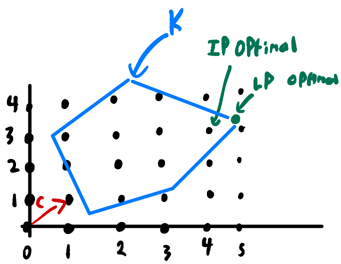
Given c, find $x \in K \cap \mathbb{Z}^n$ s.t. $\langle c,x \rangle = OPT_{IP}(c)$ (or report $K \cap \mathbb{Z}^n = \emptyset$ or $OPT_{IP}(c) = \infty$) is called an <u>integer program</u>

Bad news: integer programming is NP-hard

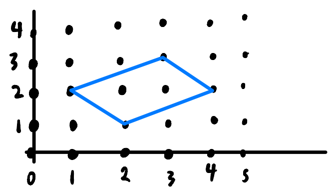
Given c, find $x \in K$ s.t. $\langle c,x \rangle = OPT_{LP}(c)$ (or report...) is called the <u>LP relaxation</u> of

Good news: $LP \in P$ → will see in later classes

Question then is how well the LP relaxation approxs the IP



(not integral)



(integral)

K is <u>integral</u> iff all BFS of K are integral ($x \in \mathbb{R}^n$ is <u>integral</u> if $x \in \mathbb{Z}^n$)

Given a set of polyhedra Π and costs $\mathcal{C}$ the <u>integrality gap</u> is

$$\max_{\substack{K \in \Pi \\ c \in \mathcal{C}}} \frac{OPT_{LP}(c)}{OPT_{IP}(c)}$$

E.g. $c = 1$ and K is all polyhedra corresponding to instances of max matching

$OPT_{LP}(c) \geq OPT_{IP}(c)$ $\forall c$ so $IG \geq 1$

Note: if all $K \in \Pi$ integral then $IG = 1$

The LP Framework

1) Express Problem as an IP
2) Get a BFS solution X to LP relaxation of (Must assume 1 exists)
3) Use X to Solve Problem (Sometimes only approximately)

↳ Simplest (3) is when K is integral b/c X is a solution to IP and so to Problem

Back to Matching

Let $K_M := \left\{ x \in \mathbb{R}^m : \begin{array}{ll} x(\delta(u)) \leq 1 & \forall u \in V \\ x_e \in [0,1] & \forall e \in E \end{array} \right\}$

So $IP_M \iff$ find $x \in K_M \cap \mathbb{Z}^m$ s.t. $\langle 1, x \rangle = IP_{OPT}(1)$

w/ LP relaxation find $x \in K_M$ s.t. $\langle 1, x \rangle = LP_{OPT}(1)$

Claim: K_M is integral

↓ gives

Algorithm for Matching

(1) [Consider IP_M

(2) [Let $y \in K_M$ be a BFS s.t. $\langle 1, y \rangle = \max_{x \in K} \langle 1, x \rangle$

(3) [Return matching corresponding to y

Always exists b/c
1) LPOPT $\neq \infty$
2) $x \geq 0$ in constraints

So far need exp. time but will see a poly-time alg.

Proof that K_M is integral

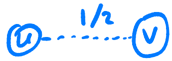
Let y be a BFS of K_M

Say $e \in E$ is fractional if $y_e \in (0,1)$, let $F := \{e \in E : y_e \in (0,1)\}$, AFSOC $|F| \geq 1$

Say vertex $u \in V$ is tight iff $y(\delta(u)) = 1$

$\forall \{u,v\} \in E$ fractional, either u or v or both are tight

AFSOC $e = \{u,v\}$ fractional but u and v not tight

e.g. 

Let $\varepsilon = \operatorname{Min}\left(1 - y(\delta(u)), 1 - y(\delta(v)), y_e\right) \rightarrow$ Note $\varepsilon > 0$

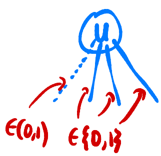
Let $\chi_e := (0, 0, \ldots, 1, \ldots, 0, 0) \in \mathbb{R}^m$ and let $y^+ := y + \varepsilon \cdot \chi_e$ and $y^- = y - \varepsilon \chi_e$

Position of e

Then $y^-, y^+ \in K_M$, $y^- \neq y^+$ but $\frac{1}{2} \cdot y^+ + \frac{1}{2} \cdot y^- = y$, contradicting y an extreme point

$y^- \in K_M$ b/c $\varepsilon < \chi_e$, $y^+ \in K_M$ b/c $y^+(\delta(u)) = y(\delta(u)) + \varepsilon \leq 1$

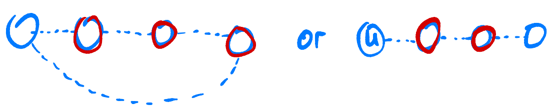
If $\{u,v\}$ is fractional and u is tight, then $|\delta(u) \cap F| \geq 2$



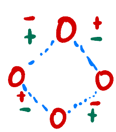
$\rightarrow$ B/C $1 = y(\delta(u)) = \sum\limits_{e \in \delta(u)} y_e$ and can't $\sum$ exactly one $\# \in (0,1)$ w/ other $\#s \in \{0,1\}$ to get 1

$(L \sqcup R, F)$ either has a cycle or a path $(u, \ldots, v)$ where u and v not tight

Start at a fractional edge and follow edges out of tight endpoints



Suppose a cycle C



Must have even # edges

b/c bipartite $\rightarrow$ each vtx. inc. to 1 even, 1 odd edge

Let y^+ add ε to even edges, $-\varepsilon$ to odd edges

y^- add $-\varepsilon$ to even edges, ε to odd edges

For $\varepsilon = \operatorname{Min}\limits_{e \in C}(y_e, 1 - y_e) > 0$, same contradiction as before

Suppose a path P

Let y^+, y^- be as in cycle case

but w/ $\varepsilon = \operatorname{Min}\limits_{e \in P}\left(y_e, 1 - y(\delta(u)), 1 - y(\delta(v))\right)$

Same contradiction as before $\left(y^+, y^- \in K_M, y^+ \neq y^- \text{ and } \frac{1}{2} \cdot y^+ + \frac{1}{2} y^- = y\right)$

Bonus: Can solve max weight bipartite matching in same way

Given bipartite graph $G=(L \cup R, E)$ $w: E \to \mathbb{R}$ find a matching $M \subseteq E$ maximizing $\sum_{e \in M} w(e)$

Same argument, just replace 1 w/ w

Other Common Integral LPs

- Flow (on hw) → generalized by TU matrix + network matrix LPs
- Cut (on hw)
- MST → generalized by matroid LPs
- General graph matching (on hw)
- Arborescence (directed MST) → generalized by matroid intersection LPs
- Matroid
- Matroid Intersection
- ...