Today

1) LPs as "shapes"

2) Geometry of Inner Products

3) Basics of Convex Geometry

4) LPs as Convex Shapes

Recall

$x \in K$ is a <u>basic feasible solution (BFS)</u> iff $\text{rank}(\text{Tight}(x)) = n$

<u>Corollary:</u> Given $x \in K = \{x : Ax \leq b\}$ w/ $T := \text{Tight}(x)$ and $w \in \text{Ker}(A_T)$
$\exists \delta > 0$ s.t. $x \pm \delta \cdot w \subseteq K$

<u>Claim:</u> If y is a BFS of $\{x : Ax \leq b\}$
then it is the unique soln. to
$$Bx = b_B$$
$\forall B \subseteq \text{Tight}(y)$ a basis of $\mathbb{R}^n$

LPs as a "Shape"

$$K = \{x : Ax \le b\} = \{x : \langle a_i, x\rangle \le b_i \ \ \forall\, a_i \in \text{rows}(A)\}$$

A bunch of points in $\mathbb{R}^n$

What kind of "shape"?

A set of the form $K = \{x : Ax \le b\}$ is called a <u>Polyhedron</u>

If $\exists\, a \in \mathbb{R}$ s.t. $d(x,y) \le a \ \ \forall x, y \in K$, then K is called a <u>Polytope</u>

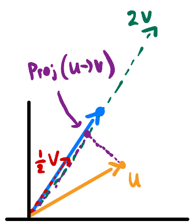
↳ K is "bounded"

Inner Products + Friends

Geometry of $\langle u, u\rangle$: $\sqrt{\langle u, u\rangle} := \sqrt{\sum_i (u_i)^2} = d(0, u) = \text{length of } u$

↳ Notated $\|u\|$

Geometry of $\langle u, v\rangle$: $\langle u, v\rangle := \sum_i u_i v_i = \ ?$



What $c \in \mathbb{R}$ minimizes $d(c \cdot v, u)$?

Let $c^* := \arg\min_c d(c \cdot v, u)$ and $\text{Proj}(u \to v) := c^* \cdot v$

Do some calculus and get $c^* = \frac{\langle u, v\rangle}{\langle v, v\rangle}$

$\min_c d(c \cdot v, u) = \min_c \left(d(c \cdot v, u)\right)^2$ b/c $d \ge 0$ and squaring is monotone increasing

So let $f(c) = \left(d(cv, u)\right)^2 = \sum_j (c \cdot v_j - u_j)^2 = \sum_j c^2 v_j^2 - 2c \cdot v_j u_j + u_j^2$

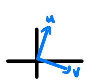
$\frac{df}{dc} = 2\sum_j c v_j^2 - 2 v_j u_j$

Set $\frac{df}{dc} = 0$, solve for c to get $c = \frac{\langle u, v\rangle}{\langle v, v\rangle}$

Then do 2nd derivative test

$\text{Proj}(u \to v) = \langle u, v\rangle \frac{v}{\langle v, v\rangle}$

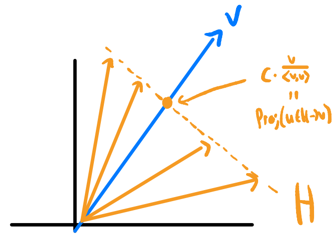
So $\langle u, v\rangle$ gives how much u is in direction of v (up to normalizing by $\langle v, v\rangle$)

Notice, $\langle u, v\rangle = 0 \rightarrow$  $\Rightarrow u, v$ orthogonal

Given $v \in \mathbb{R}^n$, $c \in \mathbb{R}$ $\quad H_= := \{u : \langle u, v \rangle = c\}$ is called a <u>hyperplane</u>

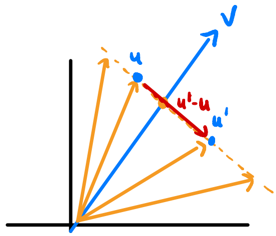
Equivalently, $\quad H_= = \{u : \mathrm{Proj}(u \to v) = c \cdot \frac{v}{\langle v, v \rangle}\}$

↳ so $u, u' \in H \to \mathrm{Proj}(u \to v) = \mathrm{Proj}(u' \to v)$

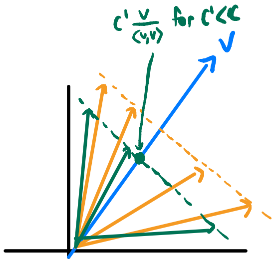


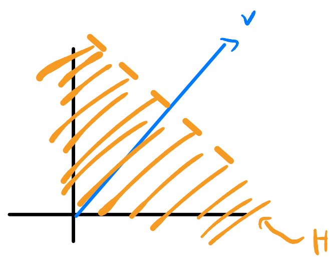
v is called the <u>normal vector</u> of H

B/c if $u, u' \in H$ then $\langle u' - u, v \rangle = \langle u', v \rangle - \langle u, v \rangle = c - c = 0$

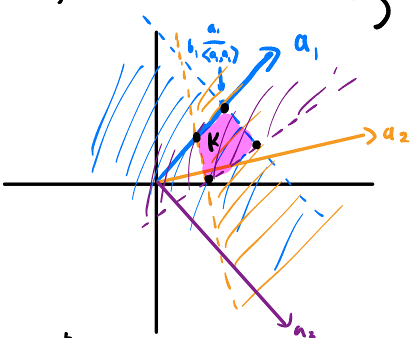


Given $v \in \mathbb{R}^n$, $c \in \mathbb{R}$ $\quad H := \{u : \langle u, v \rangle \leq c\}$ is called a <u>halfspace</u>





$K = \{x : \langle a_i, x \rangle \leq b_i \quad \forall a_i \in \mathrm{rows}(A)\}$ is the intersection of m halfspaces



Notice if x is tight for

$\langle a_i, x \rangle \leq b_i$

then x is in the hyperplane $\{x : \langle a_i, x \rangle = b_i\}$

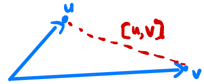
Also notice if only 2 constraints here get polyhedra

Given $K \subseteq \mathbb{R}^n$, $x \in K$ is a <u>vertex</u> if $\exists\, w$ s.t. $\langle w, x \rangle > \langle w, y \rangle$ $\forall y \in K, y \neq x$

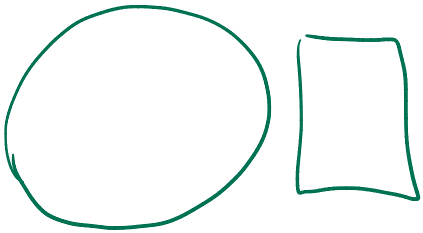
Intro to Convex Geometry

Given $u,v \in \mathbb{R}^n$, $p \in [0,1]$, $w = p\cdot u + (1-p)\cdot v$ is called a <u>convex combination</u> of u,v

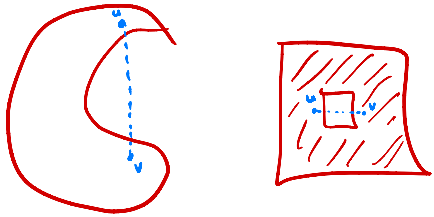
$$[u,v] := \{p\cdot u + (1-p)\cdot v \,;\, p \in [0,1]\}$$



$K \subseteq \mathbb{R}^n$ is <u>convex</u> if $u,v \in K \rightarrow [u,v] \subseteq K$



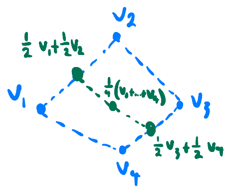
Convex



Not convex

Given $V = \{v_1, v_2, \ldots\} \subseteq \mathbb{R}^n$, the <u>convex hull</u> of V is

$$\mathrm{Con}(V) := \left\{ \sum_i p_i \cdot v_i \,:\, p_i \in [0,1]\ \forall i,\ \sum_i p_i = 1 \right\}$$



<u>Claim</u>: $\mathrm{Con}(V)$ is convex

Consider $u, v \in \mathrm{Con}(V)$ and $p \in [0,1]$

$u = \sum_i v_i p_i$ and $v = \sum_i v_i p_i'$

So $p\cdot u + (1-p)\cdot v = p \cdot \sum_i v_i p_i + (1-p) \sum_i v_i p_i' = \sum_i v_i (p\cdot p_i + (1-p) p_i')$

Also $(p \cdot p_i + (1-p) p_i') \in [0,1]$ and $\sum_i (p\cdot p_i + (1-p)p_i') = p \cdot \sum_i p_i + (1-p) \sum_i p_i' = p + (1-p) = 1$

Skippable

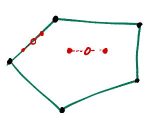
<u>Claim:</u> If K is convex and $V \subseteq K$ then $\mathrm{Con}(V) \subseteq K$ → "$\mathrm{Con}(V)$ is smallest convex set containing V"

Consider $w = v_1 p_1 + v_2 p_2 + \ldots + v_m p_m$; WTS if $V \subseteq K$ and K convex then $w \in K$; induct on m

BC: $m = 1 \rightarrow w = v_i$ for some i and $v_i \in V \subseteq K$

IS: $m \geq 2 \rightarrow w = v\, p + (1-p)\cdot v_m$ where $p = \sum_{i=1}^{m-1} p_i$ and $v = \frac{1}{p} \sum_{i=1}^{m-1} p_i v_i$; $v \in K$ by IH so $w \in K$ by convexity of K

Given $K \subseteq \mathbb{R}^n$, x is an <u>extreme point</u> of K if $x \in [y,z]$ for $y,z \in K \rightarrow y = z$



Fact: Polyhedron $K = \{Ax \leq b\}$ is convex

Suppose $u, v \in K$ and $p \in [0,1]$

$A(p \cdot u + (1-p) \cdot v) = p\, A(u) + (1-p) \cdot A(v) \leq p \cdot b + (1-p) \cdot b \leq b$

So $p \cdot u + (1-p) v \in K$ so $[u,v] \subseteq K$ so K convex

Fact: $x \in K = \{Ax \leq b\}$ is a BFS iff it is a vertex iff it is an extreme point

BFS $\rightarrow$ vertex

Let $x \in K$ be a BFS and fix basis B of $\mathrm{Tight}(x)$; x is unique point in K tight for B (by claim)

Let $w := \sum_{a_i \in B} a_i \rightarrow$ Since $B \subseteq \mathrm{Tight}(x)$, have $\langle x, w \rangle = \sum_{i: a_i \in B} b_i$

OTOH for any $y \in K$ s.t. $y \neq x$, have $\langle y, w \rangle < \sum_{i: a_i \in B} b_i$ (Since y not tight for all constraints $\langle a_i, y \rangle \leq b_i$, $a_i \in B$

extreme point $\rightarrow$ BFS

By contrapositive $\rightarrow$ Suppose x not a BFS so $\mathrm{rank}(\mathrm{Tight}(x)) < n$ so $\exists\, w \neq 0 \in \mathrm{Ker}(\mathrm{Tight}(x))$

But then by corollary have $\exists\, \delta > 0$ s.t. $x \pm \delta \cdot w \subseteq K$ so $x = \frac{1}{2}(x + \delta \cdot w) + \frac{1}{2}(x - \delta \cdot w)$ $\neq 0$

so x not an extreme point

vertex $\rightarrow$ extreme point

By contrapositive $\rightarrow$ Suppose x not an extreme point so $x = p \cdot u + (1-p) v$ for some $p \in (0,1)$, $u \neq v$, $u, v \in K$

For a given $w \in \mathbb{R}^n$ have $\langle w, x \rangle = p \langle w, u \rangle + (1-p) \langle w, v \rangle$

$\hookrightarrow$ Follows that either $\langle w, u \rangle \geq \langle w, x \rangle$ or $\langle w, v \rangle \geq \langle w, x \rangle$ so x not a vertex

Surprisingly hard to show

Fact: If K is a polytope w/ vertices V then $K = \mathrm{conv}(K)$

Summary

The polyhedron $K = \{x : Ax \leq b\}$ is the intersection of m halfspaces and is convex

$x \in K$ is a BFS iff vertex iff extreme point

If K is a polytope then it is the convex hull of its BFSs/vertices/extreme points