Today

1) The ellipsoid algorithm
2) Correctness assuming key lemma
3) Ellipsoids, formally
4) Sketch of key lemma

Recall: LP Feasibility Problem is decide if $K := \{Ax \le b\} = \emptyset$ (A: $m \times n$)

LP Search: return $x \in K$ or report $K = \emptyset$

LP Optimization: find $x \in K$ s.t. $\langle x, c \rangle = OPT$ or report $K = \emptyset$ or $OPT = \infty$

reduces to

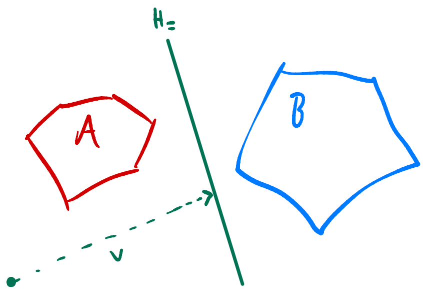
reduces to

Given $A, B \subseteq \mathbb{R}^n$, $H_= = \{u : \langle u, v \rangle = c\}$ strictly separates A and B if

$$\langle v, a \rangle < c < \langle v, b \rangle \quad \forall a \in A, b \in B$$

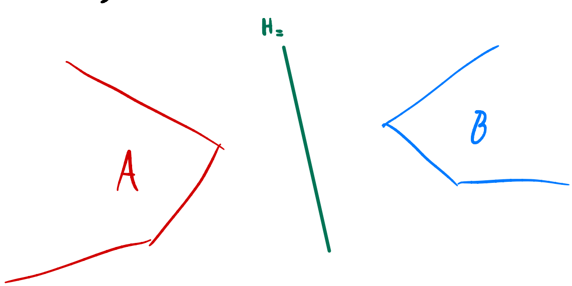
Note: suffices to find v s.t.

$$\langle v, a \rangle < \langle v, b \rangle \quad \forall a \in A, b \in B$$



Polyhedral Separation

Given disjoint non-empty polyhedra A, B, $\exists$ a hyperplane that strictly separates A, B



Goal: Decide if $K=\emptyset$ in $\text{Poly}(n)$ time

Main Takeaway: Can solve LP feasibility in Polynomial time (even w/ exponentially-many constraints)

Volume + Balls

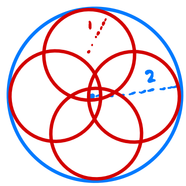
For $S\subseteq\mathbb{R}^n$, the **volume** of S is $\int_{x\in S} 1\,dx$

The (closed) radius r **ball** centered at $x\in\mathbb{R}^n$ is

$$B(x,r):=\{y\in\mathbb{R}^n : d(x,y)\le r\}$$

Let $V_n := \text{Vol}(B(x,1))$

Fact: $\text{Vol}(B(x,r)) = V_n\cdot r^n$ $\forall r\ge 0, x\in\mathbb{R}^n$

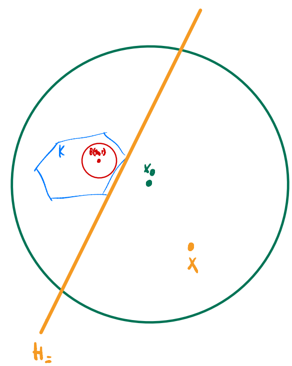


Assumptions

1) Given x_0, R s.t. if $K\neq\emptyset$ have $K\cap B(x_0,R)\neq\emptyset$

2) $\exists\, y_0, r<R$ s.t. if $K\neq\emptyset$ $B(y_0,r)\subseteq K$ w/ $\frac{R}{r}\le\exp(\text{Poly}(n))$

3) Have a "Strong Separation oracle" for K (in $\text{Poly}(n)$ time)



Given $x\in\mathbb{R}^n$, a **strong separation oracle** for $K\subseteq\mathbb{R}^n$ either correctly outputs "$x\in K$" or returns an $a_i\in\text{rows}(A)$ s.t. $H_=:=\{x:\langle a_i,x\rangle=b_i\}$ strictly separates $\{x\}$ from K

Ellipsoid Algorithm

Let $E_0 := B(v,R)$

For $i=0,1,2,\dots, 10\cdot n^2\cdot\ln(\frac{R}{r})$

- Let $c_i := \text{center}(E_i)$
- If $c_i\in K$, return $K\neq\emptyset$
- O/w let $a_i\in\text{rows}(A)$ satisfy $\langle a,x\rangle < \langle a_i, c_i\rangle$ $\forall x\in K$

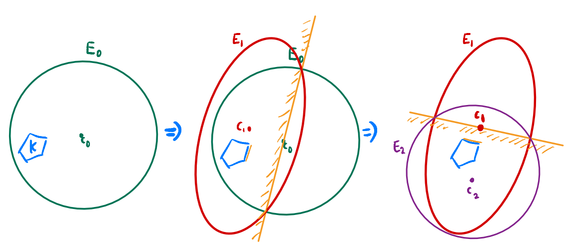
Via separation oracle

- Let E_{i+1} be a carefully chosen ellipsoid containing $E_i\cap\{x:\langle a_i,x\rangle\le\langle a_i,c_i\rangle\}$

Will be min volume ellipsoid

Notice this is through center c_i

Return $K=\emptyset$



Key Lemma : Let $T_i := E_{i-1} \cap \{x : \langle a_i, x\rangle \leq \langle a_i, c_{i-1}\rangle\}$

Then $T_i \subseteq E_i$ and $\mathrm{vol}(E_i) \leq \left(1 - \frac{1}{5n}\right) \cdot \mathrm{vol}(E_{i-1})$

Thm: Given (1)-(3), can solve LP feasibility in Poly(n) time

Proof of Ellipsoid Correctness → A halving argument!

If $K = \emptyset$ then never return $K \neq \emptyset$

If $K \neq \emptyset$, know $\exists r, y_0$ as described, AFSOC returned $K = \emptyset$

$\mathrm{vol}(K) \geq V_n \cdot r^n$ (since $B(y_0, r) \subseteq K$)

Also, by separation oracle guarantees, $K \subseteq E_i \ \forall i$ so $\forall i$

so $\mathrm{Vol}(E_i) \geq \mathrm{Vol}(K) \geq V_n \cdot r^n$

Let $k := 10 \cdot n^2 \cdot \ln\left(\frac{R}{r}\right)$ be # of iterations and AFSOC

$\mathrm{Vol}(E_0) = V_n \cdot R^n$ and by claim have

$$\mathrm{Vol}(E_k) \leq \left(1 - \frac{1}{5n}\right) \cdot \mathrm{Vol}(E_{k-1}) \leq \left(1 - \frac{1}{5n}\right)^2 \cdot \mathrm{Vol}(E_{k-2}) \leq \dots \leq \mathrm{Vol}(E_0) \cdot \left(1 - \frac{1}{5n}\right)^k$$

$$= V_n \cdot R^n \left(1 - \frac{1}{5n}\right)^k$$

$$\leq V_n \cdot R^n \exp(-k/5n)$$

$$= V_n \cdot R^n \cdot \exp\left(-2 \cdot n \cdot \ln\left(\frac{R}{r}\right)\right)$$

$$= V_n \cdot R^n \cdot \left(\frac{r}{R}\right)^{2n}$$

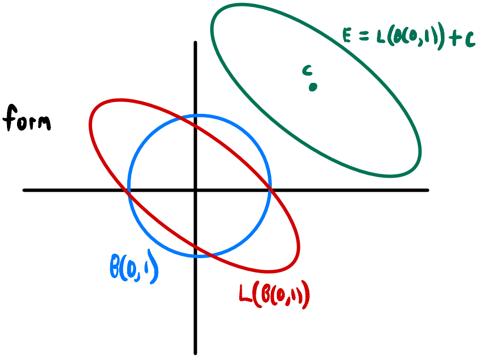
$$= V_n \cdot \left(\frac{r}{R}\right)^n \cdot r^n$$

$$< V_n \cdot r^n$$ → contradicts $\mathrm{Vol}(E_k) > V_n \cdot r^n$

Proof of Ellipsoid Runtime

Runtime is $T_{\text{separation}} \cdot O\left(n^2 \cdot \ln\left(\frac{R}{r}\right)\right) = \mathrm{Poly}(n)$

Ellipsoids



$E \subseteq \mathbb{R}^n$ is an <u>ellipsoid</u> w/ center c iff it is of form

$$E = \{L(x)+c : x \in B(0,1)\}$$

where $L: \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear fn. and $c \in \mathbb{R}^n$

<u>Fact:</u> The volume of S is $V_n \cdot |\det(L)|$

Hard to tell if $x \in E$ from this dfn., work backwards to nicer form

Matrix Q is <u>positive definite</u> if it is of form LL^T for L invertible

<u>Fact:</u> $E \subseteq \mathbb{R}^n$ is an ellipsoid w/ center c iff it is of form

a "quadratic form"

$$E = \{y : (y-c)^T Q^{-1} (y-c) \le 1\}$$

Where Q is positive definite $\to$ Notate $E := E(c, Q)$

Skippable

Consider ellipsoid E w/ $c=0$

defined since L invertible

$$E = \{y : L^{-1}y \in B(0,1)\}$$

$$= \{y : \|L^{-1}y\|^2 \le 1\}$$ (Since $L^{-1}x \in B(0,1) \to \|L^{-1}x\| \le 1 \to \|L^{-1}x\|^2 \le 1$)

$$= \{y : (L^{-1}y)^T (L^{-1}y) \le 1\}$$ ($\|\cdot\|$ definition)

$$= \{y : y^T (L^{-1})^T (L^{-1}y) \le 1\}$$ ($(Ax)^T = x^T A^T$ always)

$$= \{y : y^T ((L^{-1})^T L^{-1}) y \le 1\}$$ (commutativity of matrix mult.)

$$= \{y : y^T (LL^T)^{-1} y \le 1\}$$

$\left((A^{-1})^T = (A^T)^{-1} \text{ if } A \text{ square invertible b/c} \quad A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I \quad (A^{-1})^T A^T = (AA^{-1})^T = I^T = I \right)$

Next, consider E w/ center $c \ne 0$

Have $E = \{y + c : L^{-1}y \in B(0,1)\}$

By previous part

$$= \{y + c : y^T Q^{-1} y \le 1\} = \{y' : (y'-c)Q^{-1}y' \le 1\}$$

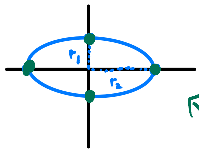
By $y' = y - c$

Claim: If $E = \{ y : (y-c)^T Q^{-1} (y-c) \le 1 \}$ then $\mathrm{Vol}(E) = \sqrt{\det(Q)} \cdot V_n$

Let $Q = LL^T$. Know $\mathrm{Vol}(E) = \det(L)$.
Also $\det(L) = \det(L^T)$ (always) so $\det(Q) = \det(L \cdot L^T) \underset{\uparrow}{=} \mathrm{Vol}(E)^2$

Since L square

Example: if $Q = \begin{pmatrix} r_1^2 & & 0 \\ & r_2^2 & \\ 0 & & \ddots \ r_n^2 \end{pmatrix}$ then $E(0,Q)$ is ellipsoid w/ "radius" r_i in i direction



B/c then $Q^{-1} = \begin{pmatrix} r_1^{-2} & & 0 \\ & r_2^{-2} & \\ 0 & & \ddots \ r_n^{-2} \end{pmatrix}$

so $y = (0, \dots, 0, x_i, 0, \dots 0)$ iff $x_i^2 \cdot r_i^{-2} \le 1$ iff $|x_i| \le r_i$

Defining E_i

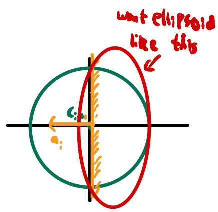
Key Lemma: Let $T_i := E_{i-1} \cap \{ x : \langle a_i, x \rangle \le \langle a_i, c_{i-1} \rangle \}$

Then $T_i \subseteq E_i$ and $\mathrm{vol}(E_i) \le \left(1 - \frac{1}{5n}\right) \cdot \mathrm{vol}(E_{i-1})$

Easy case: $E_{i-1} = E(0, I) = B(0,1)$ and $a_i = (-1, 0, 0, \dots) = -e_1$

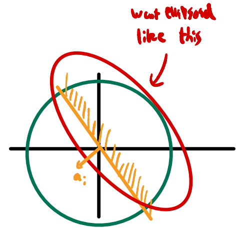
Slightly "squishes" x_1 coordinate to <1

Let $c_i = (\frac{1}{n+1}, 0, 0, \dots)$ and $Q_i := \frac{n^2}{n^2 - 1} \begin{pmatrix} \frac{n-1}{n+1} & & 0 \\ & 1 & \\ 0 & & 1 & \\ & & & 1 \end{pmatrix}$



$T_i := B(0,1) \cap \{ x : \langle a_i, x \rangle \le \langle a_i, c_{i-1} \rangle \} = B(0,1) \cap \{ x : x_1 \ge 0 \}$

Medium Case: $E_{i-1} = E(0, I) = B(0,1)$ and a_i arbitrary

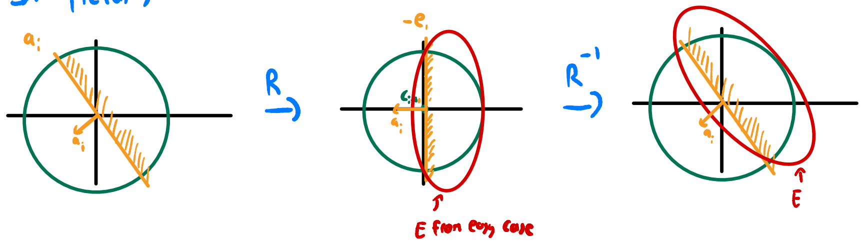


Intuition: just rotate to other case

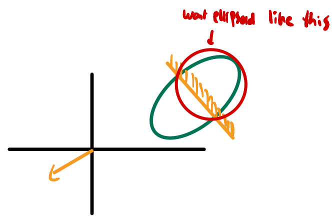
Claim: $\exists$ poly-time computable invertible linear function R (a rotation) s.t. $R(a_i) = -e_1$, $R(B(0,1)) = R(B(0,1))$

Proof assuming Claim + easy case

In pictures

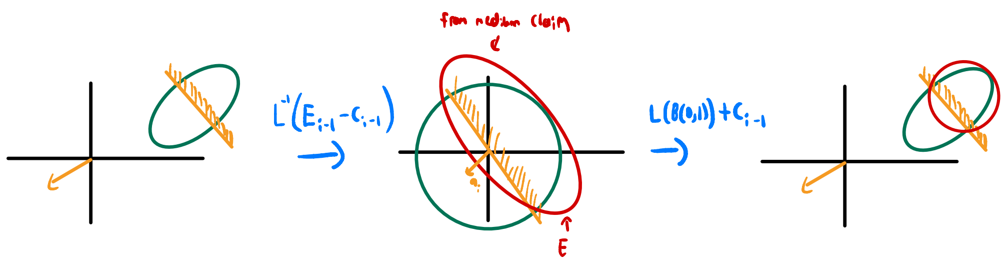


General Case: E_{i-1} and a_i arbitrary



Since E_{i-1} is an ellipsoid $\exists$ invertible L s.t. $L(B(0,1)) = E_{i-1} - c_{i-1}$

In Pictures



Proof of Key Lemma in Easy Case

$T \subseteq E_i$

Consider $x \in T \to \|x\| \le 1$ and $x_1 \ge 0$ so $\|x\|^2 \le 1$ so $\|x_{2:n}\|^2 \le 1 - x_1^2$ (*)

WTS $(x - c_i)^T Q_i^{-1} (x - c_i) \le 1$

$$\text{LHS} = \left(x_1 - \frac{1}{n+1}, x_{2:n}\right)^T \frac{n^2-1}{n^2} \begin{pmatrix} \frac{n+1}{n-1} & 0 \\ 0 & \ddots \end{pmatrix} \left(x_1 - \frac{1}{n+1}, x_{2:n}\right)$$

$$= \underbrace{\frac{n^2-1}{n^2} \cdot \left(x_1 - \frac{1}{n+1}\right)^2 \cdot \frac{n+1}{n-1}} + \frac{n^2-1}{n^2} \underbrace{\|x_{2:n}\|^2}_{\le 1 - x_1^2 \text{ (by (*))}}$$

(by expanding quadratic form)

$$= \frac{(n+1)^2}{n^2}\left(x_1 - \frac{1}{n+1}\right)^2$$

$$= \frac{1}{n^2}\left(x_1^2 (n+1)^2 - 2x_1(n+1) + 1\right)$$

$$= \frac{1}{n^2}\left(x_1^2 (n+1)^2 - 2x_1(n+1) + 1 \quad + \quad n^2 - 1 - x_1^2 (n-1)(n+1)\right)$$

$$= \frac{1}{n^2}\left((n+1) x_1^2 (n+1 - n + 1) - 2(n+1) x_1 + n^2\right)$$

$$= \frac{1}{n^2}\left(2(n+1)(x_1^2 - x_1) + n^2\right) \le 1 \quad \left(\text{by } x_1 \in [0,1] \to x_1^2 - x_1 \le 0\right)$$

$$\text{vol}(E_i) \le \left(1 - \frac{1}{5n}\right) \text{vol}(E_{i-1})$$

Suffices to show $\text{vol}(E_{i-1}) \le \left(1 - \frac{1}{5n}\right)$ Since $\text{vol}(E_{i-1}) = 1$

$$\text{vol}(E_i) = \sqrt{\det(Q)}$$

$$= \sqrt{\frac{n-1}{n+1}\left(\frac{n^2}{n^2-1}\right)^n} \quad \text{(by det of diagonal matrix is } \Pi \text{ of entries)}$$

$$= \sqrt{\frac{n+1-2}{n+1}\left(\frac{n^2-1+1}{n^2-1}\right)^n} = \sqrt{\left(1 - \frac{2}{n+1}\right)\left(1 + \frac{1}{n^2-1}\right)^n}$$

$$\le \exp\left(-\frac{1}{n+1} + \frac{n/2}{n^2-1}\right) \quad \left(\text{by } 1 + x \le e^x\right)$$

$$= \exp\left(\frac{-(n-1) + n/2}{n^2-1}\right) = \exp\left(\frac{-n/2 + 1}{n^2-1}\right) = \exp\left(-\frac{n-2}{2n^2-2}\right)$$

$$\le 1 - \frac{n-2}{4n^2-4} \quad \left(\text{by } e^{-x} \le 1 - .5x \text{ for } x \in (0,1)\right)$$

$$\le 1 - \frac{1}{5n} \quad \text{(for } n \text{ large enough)}$$

To Verify

Let $\hat{a}_i := \frac{a_i}{\|a_i\|} + e_1$

Let $R := I - 2 \cdot \frac{\hat{a}_i \otimes \hat{a}_i}{\|\hat{a}_i\|}$

Proof of Medium Case Claim

Proof of Medium Case in Calculation

Proof of General Case in Calculation

Dispensing w/ Assumptions (1)+(2)

A Proper Definition of Volume