Today

- LP Feasibility Duality (Farkas lemma) ②
- LP Optimization Duality ③
- E.g. of max flow
- Hyperplane Separation Theorems ①
- ① → ②
- ② → ③
- Proof of ①

Recall

Q1: Given function $B : \mathbb{R}^n \to \mathbb{R}$ and $b \in \mathbb{R}$, $\exists x \in \mathbb{R}^n$

$$B(x) \leq b$$

Only if $\exists x$ s.t.

$$\lambda \cdot B(x) \leq \lambda \cdot b \qquad \forall \lambda \geq 0$$

Q2: Given functions $B_1, B_2, \dots, B_m : \mathbb{R}^n \to \mathbb{R}$, $b_1, b_2, \dots, b_m \in \mathbb{R}$, $\exists x \in \mathbb{R}^n$ s.t.

$$B_i(x) \leq b_i \qquad \forall i \in [m] \quad (*)$$

Only if $\exists x$ s.t. $(*)$ and $\sum_i B_i(x) \leq \sum_i b_i$

Feasibility Duality

Feasibility Goal: give certificate of non-feasibility

$\exists x$ s.t.

$$\langle a_1, x\rangle \le b_1$$
$$\langle a_2, x\rangle \le b_2$$
$$\dots$$
$$\langle a_m, x\rangle \le b_m$$

$\Updownarrow$ iff

$$\lambda_1\langle a_1, x\rangle \le \lambda_1 \cdot b_1 \qquad \forall \lambda_1 \ge 0$$
$$\lambda_2\langle a_2, x\rangle \le \lambda_2 \cdot b_2 \qquad \forall \lambda_2 \ge 0$$
$$\dots$$
$$\lambda_m\langle a_m, x\rangle \le \lambda_m \cdot b_m \qquad \forall \lambda_m \ge 0$$

$\Downarrow$ Only if

$$\left\langle \sum_i \lambda_i a_i, x\right\rangle \le \sum_i \lambda_i b_i \qquad \forall \lambda \in \mathbb{R}^m_{\ge 0}$$

$\exists x$ s.t. $Ax \le b$

$\Downarrow$ Only if

$$\langle A^T\lambda, x\rangle \le \langle \lambda, b\rangle \qquad \forall \lambda \in \mathbb{R}^m_{\ge 0}$$

Suppose $\exists \lambda \in \mathbb{R}^m_{\ge 0}$ s.t. $A^T\lambda = 0$ and $\langle \lambda, b\rangle < 0$

Then $\forall x \in \mathbb{R}^n$ have $\langle A^T\lambda, x\rangle = \langle 0, x\rangle = 0 > \langle \lambda, b\rangle$ $\rightarrow$ contradicts

I.e. $\nexists x$ s.t. $Ax \le b$

Lem: If $\exists$ $\lambda \in \mathbb{R}^m_{\ge 0}$ s.t. $A^T\lambda = 0$ but $\langle \lambda, b\rangle < 0$ then $\nexists x$ s.t. $Ax \le b$

iff version

Farkas Lemma: $\exists$ $\lambda \in \mathbb{R}^m_{\ge 0}$ s.t. $A^T\lambda = 0$ and $\langle \lambda, b\rangle < 0$ iff $\nexists x$ s.t. $Ax \le b$

Many variants

Optimization Duality

Optimization Goal: give certificate of (tight) upper bound of optimal value

Consider LP

$$\max \ \langle c, x\rangle \ \text{s.t.} \ Ax \le b$$

w/ optimal value P

Suppose $\exists \lambda \ge 0$ s.t. $A^T\lambda = c$

b/c $Ax \le b \to \langle A^T\lambda, x\rangle \le \langle \lambda, b\rangle \ \forall \lambda \ge 0$ as on previous page

Then $\forall x$ s.t. $Ax \le b$ have $\langle c, x\rangle = \langle A^T\lambda, x\rangle \le \langle \lambda, b\rangle$

I.e. $P \le \langle \lambda, b\rangle$ for this λ

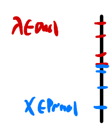
How to choose λ?

Dual LP

$\max \ \langle c, x\rangle$ s.t. $Ax \le b$ (Primal)

$\min \ \langle b, \lambda\rangle$ s.t. $A^T\lambda = c^T$, $\lambda \ge 0$ (Dual)

Let D be optimal value of dual LP



Weak Duality: If dual is feasible then $P \le D$

Proof is above

Primal infeasible → Dual unbounded

Primal unbounded → Dual infeasible

Strong Duality: If Primal feasible + bounded then $P = D$

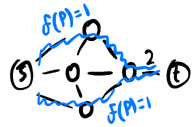
Primal + Dual Example

Maximum flow

Given graph $G=(V,E)$, capacities $u: E \to \mathbb{Z}_{\geq 0}$, $s,t \in V$

Let $\mathcal{P}(s,t)$ be all s-t paths

Want $f: \mathcal{P}(s,t) \to \mathbb{R}_{\geq 0}$ maximizing $|f| := \sum_P f_P$ s.t. $\forall e \in E$ $\sum_{P: e\in P} f_P \le u(e)$



Max flow as an LP

Variable x_P $\forall P \in \mathcal{P}(s,t)$ so $x \in \mathbb{R}^k$ where $k := |\mathcal{P}(s,t)|$

$$\max \sum_P x_P \text{ s.t.}$$

$$\sum_{P: e \in P} x_P \le u(e) \quad \forall e \in E$$

$$x_P \geq 0 \quad \forall P \in \mathcal{P}(s,t)$$

$$\Updownarrow$$

$$\max \langle c, x \rangle \text{ s.t. } Ax \le b$$

$$\Updownarrow$$

$$\max \langle \underset{c}{\mathbb{1}}, x \rangle \text{ s.t.}$$

$$\begin{array}{cc} & \begin{array}{ccccc} P_1 & P_2 \dots & P_j & \dots & P_k \end{array} \\ \begin{array}{c} y_1 \;\; e_1 \\ \vdots \\ y_i \;\; e_i \\ \vdots \\ y_m \;\; e_m \\ \lambda_1 \;\; P_1 \\ \lambda_2 \;\; P_2 \\ \vdots \\ \lambda_k \;\; P_k \end{array} & \underbrace{\begin{pmatrix} & \boxed{\mathbb{1}(e_i \in P_j)} & \\ -1 & & \\ & -1 & 0 \\ 0 & \ddots & \\ & & -1 \end{pmatrix}}_{A} \end{array} \begin{pmatrix} x_{P_1} \\ \vdots \\ \\ x_{P_k} \end{pmatrix} \le \underbrace{\begin{pmatrix} u(e_1) \\ u(e_2) \\ \vdots \\ u(e_m) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{b}$$

Max Flow LP Dual

$$\min \sum_e y_e \cdot u(e) \text{ s.t.}$$

$$\sum_{e \in P_i} y_e - \lambda_i = 1 \quad \forall P_i \in \mathcal{P}(s,t) \quad \longleftrightarrow$$

$$\lambda, y \geq 0$$

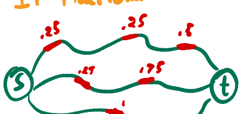
"cost" of cutting edge e

$$\min \sum_e y_e \cdot u(e) \text{ s.t.}$$

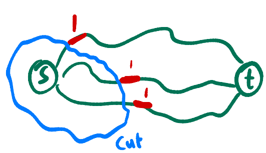
$$\sum_{e \in P_i} y_e \geq 1 \quad \forall P_i \in \mathcal{P}(s,t)$$

$$y \geq 0$$

If fractional



If integral



On HW will see

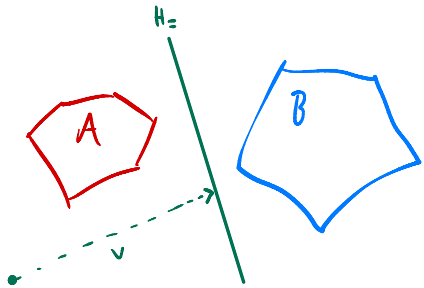
This LP is integral and has val. = Min s-t cut so max flow = Min cut

Hyperplane Separation

Given $A, B \subseteq \mathbb{R}^n$, $H_= = \{u : \langle u, v\rangle = c\}$ **strictly separates** A and B if

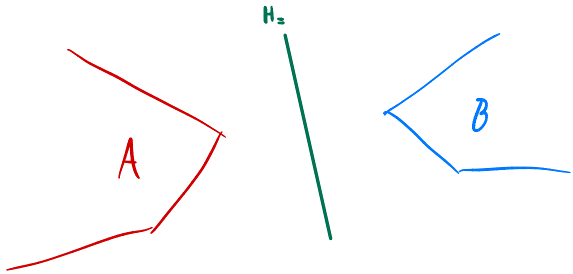
$$\langle v, a\rangle < c < \langle v, b\rangle \quad \forall a \in A, b \in B$$

Note: suffices to find v s.t. $\langle v, a\rangle < \langle v, b\rangle \quad \forall a \in A, b \in B$



Polyhedral Separation

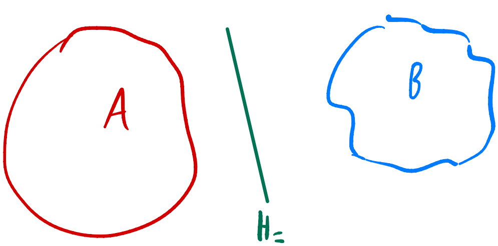
Given disjoint non-empty polyhedra A, B, $\exists$ a hyperplane that strictly separates A, B



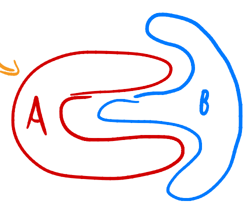
Convex Separation

Note: not immediate from above b/c polyhedra can be unbounded

Given disjoint, non-empty $A, B \subseteq \mathbb{R}^n$ convex (A closed, B compact) $\exists$ a hyperplane $H_=$ that strictly separates A, B



Not true if not convex



Proof of Convex Separation

Given A (closed), B (compact) both convex

Let $a_0 \in A$ and $b_0 \in B$ be 2 points minimizing $d(a,b)$ ← Euc. distance

→ Proof $a_0, b_0 \exists$
Let $\Delta := \inf_{a \in A, b \in B} d(a,b)$
For $b \in B$, let $f(b) := \inf_{a \in A} d(a,b)$

Since f continuous, B compact, f attains min @ some a_0
Let $\tilde{B}$ be $B \cap \text{Ball}(a_0, 2\Delta) \to \tilde{B}$ compact + $\neq \emptyset$
Let $g(b) := d(a_0, b) \to g$ continuous so min attained on $\tilde{B}$ @ b_0

Claim that $H_= := \left\{ x : \langle b_0 - a_0, x \rangle = \frac{\|b_0\|^2 - \|a_0\|^2}{2} \right\}$ Strictly separates A, B

Since $A \cap B = \emptyset$, know $b_0 - a_0 \neq 0$ so $\|b_0 - a_0\| > 0$ so $\langle b_0 - a_0, b_0 - a_0 \rangle > 0$

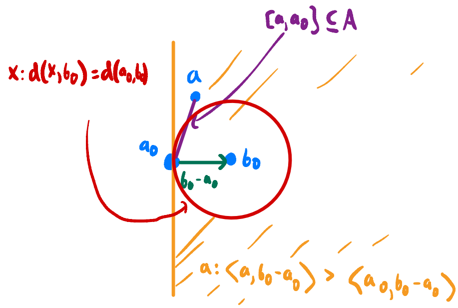
so $\langle a_0, b_0 - a_0 \rangle < \langle b_0, b_0 - a_0 \rangle$

Thus, suffices to show $\forall a \in A$ have $\langle a, b_0 - a_0 \rangle \leq \langle a_0, b_0 - a_0 \rangle$
and $\forall b \in B$ have $\langle b_0, b_0 - a_0 \rangle \leq \langle b, b_0 - a_0 \rangle$

Both cases symmetric so wts $\forall a \in A$ have $\langle a, b_0 - a_0 \rangle \leq \langle a_0, b_0 - a_0 \rangle$

AFSOC $\exists a \in A$ s.t. $\langle a, b_0 - a_0 \rangle > \langle a_0, b_0 - a_0 \rangle$

Rest of proof by Pictures:



By convexity $[a, a_0] \subseteq A$ but then $\exists x \in [a, a_0] \subseteq A$ s.t. $d(x, b_0) < d(a_0, b_0)$ $\times\!\!\times$ a_0 choice

Rest of proof by calculation:

Know $[a, a_0] \subseteq A$ by convexity; let $a_\varepsilon := (1-\varepsilon) a_0 + \varepsilon \cdot a$ $\quad (a_\varepsilon \in A \ \forall \varepsilon \in [0,1])$

$$
\begin{aligned}
\text{So} \quad d(a_\varepsilon, b_0) &= \|b_0 - (1-\varepsilon) a_0 - \varepsilon a\| = \|(1-\varepsilon)(b_0 - a_0) + \varepsilon (b_0 - a)\| \\
&= \sqrt{(1-\varepsilon)\|b_0 - a_0\|^2 + \varepsilon \|b_0 - a\|^2 + (1-\varepsilon)(\varepsilon) \cdot \langle b_0 - a_0, b_0 - a \rangle} \\
&< \sqrt{(1-\varepsilon)\|b_0 - a_0\|^2 + \varepsilon \|b_0 - a\|^2 + (1-\varepsilon)(\varepsilon) \cdot \langle b_0 - a_0, b_0 - a_0 \rangle} \\
&= \sqrt{(1-\varepsilon)\|b_0 - a_0\|^2 + \varepsilon \|b_0 - a\|^2 + (\varepsilon - \varepsilon^2) \|b_0 - a_0\|^2} \\
&= \sqrt{\|b_0 - a_0\|^2 + \varepsilon \|b_0 - a\|^2 - \varepsilon^2 \|b_0 - a_0\|^2} \\
&\leq \|b_0 - a_0\| \quad \text{for } \varepsilon \text{ sufficiently small}
\end{aligned}
$$

So $\exists \varepsilon$ s.t. $a_\varepsilon \in A$ and $d(a_\varepsilon, b_0) < d(a_0, b_0)$ $\times\!\!\times$ Choice of a_0

Proof of Polyhedral Separation w/ Convex Separation

Note if $A \subseteq \mathbb{R}^n$ is convex $\overset{+\text{ closed}}{\wedge}$ then so is $-A := \{-x : x \in A\}$ → easy to verify

Also, if A, B convex $\overset{+\text{ closed}}{\wedge}$ then $A+B := \{a+b : a \in A, b \in B\}$ is convex $\underset{\text{closed}}{+}$ → Also easy + on hw

Follows that $A-B$ is convex + closed

By $A \cap B = \emptyset$, know $0 \notin A-B$

$\{0\}$ is convex (+ compact) so by convex separation $\exists v$ s.t.

$$\langle 0, v \rangle < \langle a-b, v \rangle, \quad \forall a \in A, b \in B$$

$$\Updownarrow$$

$$\langle b, v \rangle < \langle a, v \rangle \quad \forall a \in A, b \in B \rightarrow A, B \text{ strictly separated}$$

Given $X=(\dots,x_{i-1},x_i,x_{i+1},\dots)$, $y=(\dots,x_{i-1},x_{i+1},\dots)$ is the result of Projecting out i

Let $\mathrm{Proj}(x,I)$ be the result of Projecting out all $i \in I$

"Projection of a polyhedra is a Polyhedra"

Claim: Given any $I \subseteq [n]$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $\{\mathrm{Proj}(x,I) : Ax \le b\}$ is a Polyhedra

Suffices to show for $|I|=1$ by induction; then follows by F-M elimination to eliminate x_i

Claim: Given linear $A: \mathbb{R}^n \to \mathbb{R}^m$, $I := \{Ax : x \in \mathbb{R}^n\}$ is a polyhedra → "Image of linear fn. is a Polyhedra"

Consider $I' := \{(x,Ax) : x \in \mathbb{R}^n\} \subseteq \mathbb{R}^{n+m}$; let $A' = (A \ -I)$ so $I' = \{y : A'y = 0\}$ is a Polyhedra

But $I := \{x : (x, Ax) \in I'\}$ is just the Projection of I, so follows by previous statement

Proof of Farkas w/ Polyhedral Separation

Already showed 1 direction so

WTS if $\nexists x$ s.t. $Ax \le b$ then $\exists \lambda \in \mathbb{R}^m_{\ge 0}$ s.t. $A^T\lambda = 0$ and $\langle \lambda, b\rangle < 0$

Suppose $\nexists x$ s.t. $Ax \le b$

Let $I := \{Ax : x \in \mathbb{R}^n\}$ be the image of A, $B := \{b' : b' \le b\}$

Nice as hw → I is a Polyhedra b/c it is the image of a linear fn., B is a Polyhedra b/c $B = \{b' : \langle b', 1\rangle \le b\}$

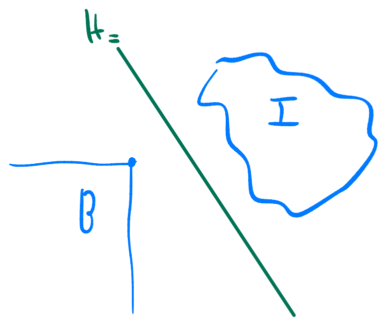
By assumption $B \cap I = \emptyset$

Also $0 \in I$ (b/c $A0 = 0$) and $b \in B$ so I, B non-empty so by Polyhedral separation

$$\exists \lambda \in \mathbb{R}^n, a \in \mathbb{R} \text{ s.t.}$$

$$\langle b', \lambda\rangle < a \quad \forall b' \in B$$

$$\langle Ax, \lambda\rangle \ge a \quad \forall x \in \mathbb{R}^n$$



$0 \in I$ so $a \le 0$ so $\langle b', \lambda\rangle < 0 \quad \forall b' \in B$

Consider $c \to \infty$

$$\forall x \in \mathbb{R}^n \quad \langle A(c\cdot x), \lambda\rangle \ge a \to \langle Ax, \lambda\rangle \ge \frac{a}{c} = 0$$

So $\forall b' \in B \quad \langle b', \lambda\rangle < 0$ and $\forall x \in \mathbb{R}^n \quad \langle Ax, \lambda\rangle \ge 0$

Lastly $\lambda \ge 0$

AFSOC $\exists$ i s.t. $\lambda_i < 0$

Then $\langle \lambda, b'\rangle > 0$ for any b' w/ $b'_i \to -\infty$, Contradicting $\langle b', \lambda\rangle < 0$

Proof of Strong Duality w/ Farkas

AFSOC that $P < D$

So $\nexists x$ s.t. $Ax \leq b$ and $\langle c, x\rangle \geq D$

$\Updownarrow$

$\nexists x$ s.t. $\underbrace{\begin{pmatrix} A \\ -c^T \end{pmatrix}}_{\hat{A}} x \leq \underbrace{\begin{pmatrix} | \\ b \\ | \\ -D \end{pmatrix}}_{\hat{b}}$

By Farkas $\exists$ $y \in \mathbb{R}^{m+1}_{\geq 0}$ s.t. $\hat{A}^T y = 0$ and $\langle \hat{b}, y\rangle < 0$

Write y as $(\underset{\in \mathbb{R}^m}{\lambda}, \underset{\in \mathbb{R}}{z})$

So $0 = \hat{A}^T y = A^T \lambda - z \cdot c^T$ and $\langle \lambda, b\rangle - D \cdot z < 0$

Have $z > 0$

AFSOC $z = 0$

Then $0 = A^T \lambda - z \cdot c^T = A^T \lambda$ so $A^T \lambda = 0$

Also $\langle \lambda, b\rangle = \langle \lambda, b\rangle - D \cdot z < 0$

So Farkas gives $\nexists$ x s.t. $Ax \leq b$, Contradicting Primal feasible

Consider $\frac{1}{z} \cdot \lambda \geq 0$

$A^T \lambda - z \cdot c^T = 0$ so $A^T \frac{\lambda}{z} = c^T$ so $\frac{1}{z} \cdot \lambda$ dual feasible

But $\langle b, \frac{1}{z} \cdot \lambda\rangle = \frac{1}{z} \cdot \langle b, \lambda\rangle < \frac{1}{z} \cdot D \cdot z = D$, Contradicting D defn.