Today

1) Algebra of Inequalities

2) LP Feasibility in doubly-exp. time (Fourier-Motzkin)

3) LP Search via Feasibility

4) LP Optimization via Search

# Algebra of Inequalities

**Q1:** Given function $B : \mathbb{R}^n \to \mathbb{R}$ and $b \in \mathbb{R}$, $\exists x \in \mathbb{R}^n$

$$B(x) \leq b$$

iff $\exists x$ s.t.

1) $B(x) + c \leq b + c \qquad \forall c \in \mathbb{R}$
2) $\lambda \cdot B(x) \leq \lambda \cdot b \qquad \forall \lambda > 0$
3) $-b \leq -B(x)$

**Q2:** Given functions $B_1, B_2, \ldots, B_m : \mathbb{R}^n \to \mathbb{R}$, $b_1, b_2, \ldots, b_m \in \mathbb{R}$, $\exists x \in \mathbb{R}^n$ s.t.

$$B_i(x) \leq b_i \qquad \forall i \in [m] \quad (*)$$

iff $\exists x$ s.t. $(*)$ but (1), (2) or (3) applied to one of the $i$

iff $\exists x$ s.t. $(*)_c$ $\leftarrow$ Let $I := \{ i : \text{"}c \leq b_i\text{"} \in (*) \}$

$\in \mathbb{R}$

$I' := \{ i : \text{"}B_i(x) \leq c\text{"} \in (*) \}$

$\bar{I} = [m] \setminus (I \cup I')$

$$(*)_i := \begin{cases} B_i(x) \leq b_i & \forall i \in \bar{I} \\ B_j(x) \leq b_i & \forall i \in I \text{ and } j \in I' \end{cases}$$

Only if $\exists x$ s.t. $(*)$ and $\sum_i' B_i(x) \leq \sum_i' b_i$

# Linear Programming Feasibility Problem

**Goal:** Q2 s.t. each $B_i(x)$ of form $\langle a_i, x\rangle$ for some $a_i \in \mathbb{R}^n$

Egs

① $\exists x = (x_1, x_2, x_3)$ s.t. $5x_1 + 2x_2 + x_3 \le 10$ Yes $x = (0,0,10)$

② $\exists x$ $\quad 2x_1 - 2x_3 \le -2$ and $x_3 - x_2 \le 0$ and $x_3 - x_1 \le 0$

iff $\exists x$ $\quad 1 + x_1 \le x_3$ and $x_3 \le x_2$ and $x_3 \le x_1$ $\quad (*)$

iff $\exists x$ $\quad 1 + x_1 \le x_2$ and $1 + x_1 \le x_1$ $\quad (*)_{x_3}$

iff $\exists x$ $\quad 1 + x_1 \le x_2$ and $1 \le 0$

No

Maybe add more motivating egs →

## How to Write LP

$\exists x$ s.t. $\langle a_i, x\rangle \le b_i \;\; \forall i \iff \exists x$ s.t. 

$$\underbrace{\begin{pmatrix} - & a_1 & - \\ - & a_2 & - \\ & \vdots & \\ - & a_m & - \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \le \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \iff \exists x \text{ s.t. } Ax \le b$$

coordinate-wise

Let $K = \{x : Ax \le b\}$

**Goal:** Decide if $K \ne \emptyset$

## For Free

1) $\langle a_i, x\rangle + b_i \le \langle a_i', x\rangle + b_i'$ iff $\langle a_i - a_i', x\rangle \le b_i' - b_i$
2) $\langle a_i, x\rangle \ge b_i$ iff $\langle -a_i, x\rangle \le -b_i$
3) $\langle a_i, x\rangle = b_i$ iff $\langle a_i, x\rangle \le b_i$ and $\langle a_i, x\rangle \ge b_i$

# Fourier-Motzkin Elimination

Given $(*)$

$\times n$ times

Step 1: for each $i$, apply (1), (2) and (3) so ith inequality of form

$x_n \leq \cdot$ or $\cdot \leq x_n$ or $\cdot \leq \cdot$

$\not\ni x_n$

Step 2: $(*) \leftarrow (*)_{x_n}$ and $n \leftarrow n-1$

Return true if all inequalities of $(*)$ true; false o/w

## Runtime:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Iteration | 0 | 1 | 2 | ... | $i$ | ... | $n$ |
| # inequalities | $m$ | $\leq m^2$ | $\leq m^4$ | ... | $\leq m^{2^i}$ | ... | $\leq m^{2^n}$ |

$\rightarrow O\left(m^{2^n}\right)$

## LP Search Problem

Let $K := \{x : Ax \le b\}$

**Goal:** Output $x \in K$ or report $K = \emptyset$

**Fact:** Given $m \times n$ matrix $A$ and $b \in \mathbb{R}^m$, let $K' = \{x : Ax = b\}$

Can output $x \in K'$ or report $K'$ empty in poly-time (Gaussian Elimination)

**Obs:** $\exists\ x \in \mathbb{R}^n$ s.t. $Ax \le b$

iff

$\exists\ x^+, x^- \in \mathbb{R}^n$ s.t. $Ax^+ - Ax^- \le b$ and $x^+, x^- \ge 0$

Let $x = x^+ - x^-$ (algorithmic)

Let $x^+ = x^+$ and $x^- = x^-$ (algorithmic)

iff

$\exists\ x^+, x^- \in \mathbb{R}^n, s \in \mathbb{R}^m$ s.t. $Ax^+ - Ax^- + \underset{m \times n \text{ identity}}{I} \cdot s = b$ and $x^+, x^-, s \ge 0$

↳ Another LP: let $x' = \begin{pmatrix} x^+ \\ x^- \\ s \end{pmatrix}$ $A' = \begin{pmatrix} A & -A & \underset{m \times n}{I} \end{pmatrix}$

Find $x' \in \{\underbrace{A'x' = b, x' \ge 0}_{\text{"Equational Form"}}\}$

If can find $x' \in \mathbb{R}^{2n+m}$ s.t. $A'x' = b'$ and $x' \ge 0$ (in $T$ time)

then can find $x \in \mathbb{R}^n$ s.t. $Ax \le b$ (in $T + O(n+m)$ time)

Let $O_+ := \{x : x \geq 0\}$

Given $S \subseteq [n]$, let $K_S = \left\{x : \begin{array}{l} Ax = b \\ x_i = 0 \;\; \forall i \in S \end{array}\right\}$ → Can find $x \in K_S(A)$ w/ GE

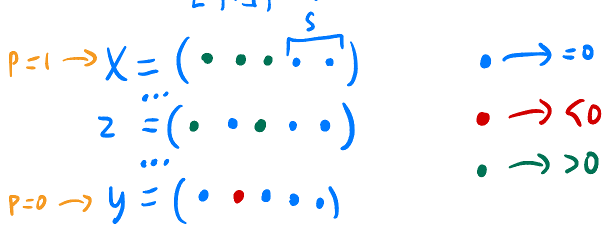
**Fact:** If $K_S \cap O_+ \neq \emptyset$ and $\exists y \in K_S \setminus O_+$

then $\exists\, i \notin S$ s.t. $K_{S+i} \cap O_+ \neq \emptyset$

Let $x \in K_S \cap O_+$

If $\exists i \notin S$ but $x_i = 0$, done (b/c $x \in K_{S+i} \cap O_+$ so $K_{S+i} \cap O_+ \neq \emptyset$)

O/w for $p \in [0,1]$, let $z := p \cdot x + (1-p) \cdot y$




Must $\exists\, p$ and $i \notin S$ s.t. $\underline{z \geq 0}$ → $z \in O_+$

and $z_j = 0 \;\forall j \in S+i$ but $A(z) = p \cdot A(x) + (1-p) A(y) = b$ → $z \in K_{S+i}$

So $z \in K_{S+i} \cap O_+$

## Search Via Feasibility

Put LP in eq. form $K := \{x : Ax = b,\ x \geq 0\}$

$S \leftarrow \emptyset$

Repeat n times:

- Do GE to compute $y \in K_S = \left\{x : \begin{array}{l} Ax = b \\ x_i = 0 \; \forall i \in S \end{array}\right\}$
- If $y \in O_+$, return $y$
- Else if $\exists i \notin S$ s.t. $K_{S+i} \cap O_+ \neq \emptyset$ ← Find: w/ feasibility alg.
  - $S \leftarrow S+i$
- O/w report $K = \emptyset$

Fork to 2nd class?
↑
Got to here in class but rushed last page

If $y$ returned, then
$y \in K_S \cap O_+ \subseteq K$ ← for some $S \subseteq [n]$

so WTS if $K \neq \emptyset$ then return a $y$

$K \neq \emptyset \to K_S \cap O_+ \neq \emptyset$ always
and
never report $K = \emptyset$

Initially $K_S \cap O_+ = K_\emptyset \cap O_+ = K \neq \emptyset$

When $y$ not returned, have

$K_S \cap O_+ \neq \emptyset$ and $y \in K_S \setminus O_+$

↑ inductively

Claim → so $\exists\, i \notin S$ s.t. $K_{S+i} \cap O_+ \neq \emptyset$

and don't ~~report~~ $K = \emptyset$

But terminate after $\leq n$ iterations

# LP Optimization Problem

get min via $-c$

Given $c \in \mathbb{R}^n$, let $K = \{x : Ax \le b\}$, let $OPT := \max_{y \in K} \langle c, y \rangle$

**Goal:** return $x \in K$ s.t. $\langle c, x \rangle = OPT$ or report $K = \emptyset$ or report $OPT = \infty$

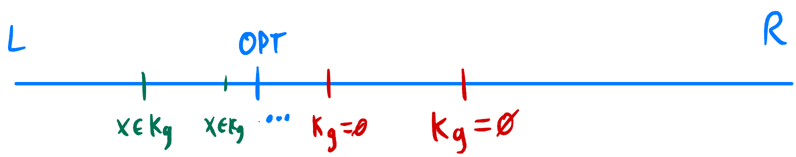
## Optimization Via Search

Assume

1) Have $L, R$ s.t. $L \le OPT \le R$, $|L - R| \le \exp(\text{poly}(n, m))$
2) $OPT \in \mathbb{Z}$

Let $K_g := \left\{ x : \begin{matrix} Ax \le b \\ \langle c, x \rangle \ge g \end{matrix} \right\}$ for $g \in \mathbb{Z}$

Binary search $g \in [L, R] \cap \mathbb{Z}$




**Caveat 1:** Possible for $OPT = \infty$

Eg. $\max x_1$ s.t. $x_1 \ge 0$

$\to$ Can poly-time compute $M \in \mathbb{R}$ s.t. if $\exists x \in K$ w/ $\langle x, c \rangle \ge M$ then $OPT = \infty$

**Caveat 2:** Where do $L, R$ come from?

$\to$ Poly-time computable

**Caveat 3:** What if $OPT \notin \mathbb{Z}$?

$\to$ After poly$(n, m)$ BS rounds, can round $g$ to $OPT$ w/ # theory in poly-time