Today

4 Series

1) nth Triangular #
2) nth Harmonic #
3) Inverse Squares
4) Geometric

Big 3 Inequalities

1) Cauchy-Schwarz
2) AM-GM
3) Jensen's Inequality (+ Convexity)

4 Series to Know

① nth Triangular Number : $\sum_{i=1}^{n} i = \Theta(n^2)$

$$\underbrace{1+2+3+\dots+n-2+n-1+n}_{n+1 \,\times \frac{n}{2} \text{ times}} = \frac{n(n+1)}{2}$$

② nth Harmonic Number : $H_n = \sum_{i=1}^{n} \frac{1}{i} = \Theta(\log n)$

$$\underbrace{\tfrac{1}{1}}+\underbrace{\tfrac{1}{2}+\tfrac{1}{3}}+\underbrace{\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}}+\underbrace{\tfrac{1}{8}+\dots}$$

$$\leq \underbrace{\tfrac{1}{1}}_{=1}+\underbrace{\tfrac{1}{2}+\tfrac{1}{2}}_{=1}+\underbrace{\tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{4}}_{=1}+\underbrace{\tfrac{1}{8}+\dots}_{=1}+\tfrac{1}{n}$$

$$\geq \underbrace{\tfrac{1}{2}}_{=\frac{1}{2}}+\underbrace{\tfrac{1}{4}+\tfrac{1}{4}}_{=\frac{1}{2}}+\underbrace{\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}}_{=\frac{1}{2}}+\dots$$

$$\Theta(\log n)$$

③ Inverse Squares : $\sum_{i=1}^{n} \frac{1}{i^2} = \Theta(1)$

Add/Subtract:
$$\frac{1}{i(i-1)} = \frac{1-i+i}{i(i-1)} = \frac{i-(i-1)}{i(i-1)} = \frac{1}{i-1} - \frac{1}{i}$$

Telescoping:
$$\sum_{i=2}^{n} \frac{1}{i-1} - \frac{1}{i} = \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \dots \cancel{\frac{1}{n-1}} - \frac{1}{n}$$
$$= 1 - \frac{1}{n}$$

$$\sum_{i=1}^{n} \frac{1}{i^2} \leq 1 + \sum_{i=2}^{n} \frac{1}{i(i-1)} = 1 + \sum_{i=2}^{n} \frac{1}{i-1} - \frac{1}{i} = 1 + 1 - \frac{1}{n} \leq 2$$

$\Omega(1)$ trivial

④ Geometric : $\sum_{i=0}^{\infty} r^i = \Theta(1)$ (for $r \in (0,1)$)

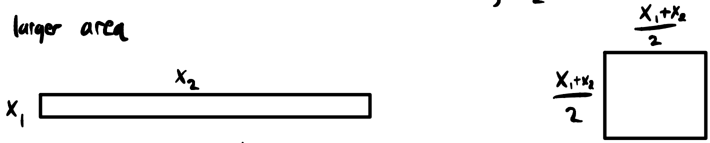
Series as Variable : Let $S := \sum_{i=1}^{\infty} r^i$ so $S = 1 + r + r^2 + \dots = 1 + r \cdot S$

so $S(1-r) = 1 \rightarrow S = \frac{1}{1-r}$

2 Geometrically "Obvious" Facts and The "Big 3" Inequalities

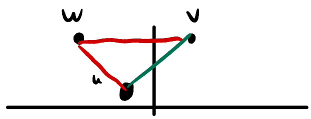
Puzzle ①: "Square maximizes area"

given a rectangle w/ side lengths x_1, x_2
then square w/ side lengths $\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}$
has larger area



Puzzle ②: "triangle inequality"

$\forall$ 3 points $u, v, w \in \mathbb{R}^2$, $u \to v \leq u \to w \to v$



Each puzzle really an algebra fact

① $$\prod_i x_i \leq \left(\frac{\sum_1^t x_i}{2}\right)^2 \iff \left(\prod_i x_i\right)^{1/2} \leq \frac{\sum_1^t x_i}{2}$$

② Recall $d(u,v) := \sqrt{\sum_i^t (u_i - v_i)^2}$

Translation invariant: $d(u+x, v+x) = d(u,v) \quad \forall x \in \mathbb{R}^2$

$$d(u,v) \leq d(u,w) + d(w,v) \quad \forall u,v,w$$

$\Updownarrow$ (TI)

$$d(0, v-u) \leq d(0, w-u) + d(0, v-w) \quad \forall u,v,w$$

$\Updownarrow$ Let $\tilde{u} = w-u$, $\tilde{v} = v-w$ $\to \tilde{u} + \tilde{v} = v-u$

$$d(0, \tilde{u}+\tilde{v}) \leq d(0,\tilde{u}) + d(0,\tilde{v}) \quad \forall \tilde{u}, \tilde{v}$$

$\Updownarrow$

$$d(0, u+v) \leq d(0,u) + d(0,v) \quad \forall u,v$$

$\Updownarrow$

$$\sqrt{\sum_i^t (u_i+v_i)^2} \leq \sqrt{\sum_i^t u_i^2} + \sqrt{\sum_i^t v_i^2} \quad \forall u,v \quad \overset{\text{rearrange}}{\iff} \quad \sum_i^t u_i v_i \leq \sqrt{\sum_i^t u_i^2 \cdot \sum_i^t v_i^2} \quad \forall u,v$$

$$\sum_i^t (u_i+v_i)^2 \leq \sum_i^t u_i^2 + \sum_i^t v_i^2 + 2\sqrt{\sum_i^t u_i^2 \sum_i^t v_i^2}$$

$\Updownarrow$

$$\sum_i^t 2u_i v_i \leq 2\sqrt{\sum_i^t u_i^2}\sqrt{\sum_i^t v_i^2}$$

Both provable w/ non-negativity of squares

↳ $x^2 \ge 0 \quad \forall x \in \mathbb{R}$

① $\sqrt{x_1 \cdot x_2} \le \dfrac{x_1 + x_2}{2}$

$\Updownarrow$

$4x_1x_2 \le x_1^2 + 2x_1x_2 + x_2^2$

$\Updownarrow$

$0 \le x_1^2 - 2x_1x_2 + x_2^2$

$\Updownarrow$

$0 \le (x_1 - x_2)^2$

② $u_1v_1 + u_2v_2 \le \sqrt{(u_1^2 + u_2^2)(v_1^2 + v_2^2)}$

$\Updownarrow$

$\cancel{u_1^2v_1^2} + 2u_1v_1u_2v_2 + \cancel{u_2^2v_2^2} \le \cancel{u_1^2v_1^2} + u_1^2v_2^2 + u_2^2v_1^2 + \cancel{u_2^2v_2^2}$

$\Updownarrow$

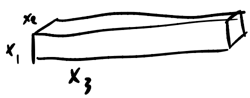
$0 \le u_1^2v_2^2 + u_2^2v_1^2 - 2u_1v_1u_2v_2$

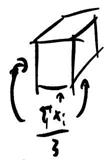
$\Updownarrow$

$0 \le (u_1v_2 - u_2v_1)^2$

Algorithms often about higher-dimensional geometry

Puzzle ①: "hypercube maximizes area"





In Algebra: $\underbrace{\left(\prod_{i=1}^{k} x_i\right)^{1/k}}_{GM} \le \underbrace{\dfrac{\sum_{i=1}^{k} x_i}{k}}_{AM} \qquad \forall x_i \in \mathbb{R}_{\ge 0}$

$\boxed{\text{AM-GM}}$

Puzzle ②': "triangle inequality (in higher dimensions)"

In Algebra: $\sum_{i=1}^{k} u_i v_i \le \sqrt{\left(\sum_{i=1}^{k} u_i\right)^2 \left(\sum_{i=1}^{k} v_i\right)^2} \qquad \forall\, u, v \in \mathbb{R}^k$

$\boxed{\text{Cauchy-Schwarz}}$

Proving Cauchy-Schwarz

By induction on k

Base Cases

$n=1$: $u_1 v_1 \le u_1 v_1$ ✓

$n=2$: Already done

Inductive Step

$$\underbrace{u_1 v_1 + u_2 v_2 + \dots + u_{k-1} v_{k-1}}_{IH} + u_k v_k \le \underbrace{\sqrt{\sum_{i=1}^{k-1} u_i^2}}_{u_1'} \underbrace{\sqrt{\sum_{i=1}^{k-1} v_i^2}}_{v_1'} + \underbrace{u_k}_{u_2'} \underbrace{v_k}_{v_2'}$$

$$\le \sqrt{(u_1')^2 + (u_2')^2} \sqrt{(v_1')^2 + (v_2')^2}$$

$$= \sqrt{\sum_1^k u_i^2} \sqrt{\sum_1^k v_i^2}$$

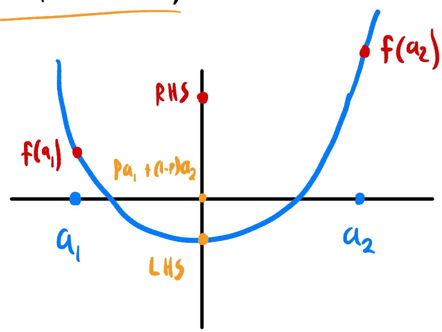
→ End of Class

Proving AM-GM

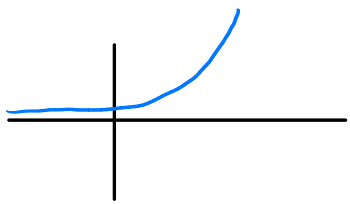
$f: \mathbb{R} \to \mathbb{R}$ is **convex** if $\forall\ p \in [0,1]$ and $a_1, a_2 \in \mathbb{R}$

$$f(pa_1 + (1-p)a_2) \le pf(a_1) + (1-p)f(a_2)$$

"Bowl shape"



Eg. e^x



Jensen's Inequality

Let $p_1, \dots, p_k$ be a probability distribution, $a_1, \dots, a_k \in \mathbb{R}$, f convex

Then $f\left(\sum_i p_i a_i\right) \le \sum_i p_i f(a_i)$ $\forall$ convex f

Proof of AM-GM w/ Jensen's

Let $a_i = \ln(x_i)$, $f = e^x$ and $p_i = \frac{1}{k}$ $\forall i$

Jensen's gives

$$e^{(\ln(x_1) + \ln(x_2) + \dots)/k} \le \sum_i \frac{1}{k} e^{\ln(x_i)}$$

So $\left(\prod x_i\right)^{1/k} \le \sum_i x_i / k$

Proving Jensen's Inequality

By induction on k

Base Case: $k=2 \to$ defn. of convexity

Inductive Step:

Let $p = \sum_{i=1}^{k-1} p_i$ so $p_k = 1-p$

$$f(p_1x_1 + \dots + p_kx_k) = f\left(p\left(\frac{p_1x_1}{p} + \dots + \frac{p_{k-1}x_{k-1}}{p}\right) + (1-p)x_k\right)$$

$$\leq p f\left(\frac{p_1x_1}{p} + \dots + \frac{p_{k-1}x_{k-1}}{p}\right) + (1-p) f(x_k) \quad \text{(IH)}$$

$$\leq p\left(\frac{p_1}{p} f(x_1) + \dots + \frac{p_{k-1}}{p} f(x_{k-1}) + p_k \cdot x_k \quad \text{(IH)}$$

$$= \sum_i p_i f(x_i)$$