Today

1) Valiant's Hypercube Routing
2) Karger's Min Cut Algorithm (Probably not time)

Concentration Paradigm

To show fact (*)

a) Show (*) true if all RVs near $\mathbb{E}$

b) Concentration: each RV at $\mathbb{E}$ ($\pm \log n$) w/ good probability

c) Union bound: all RVs

Chernoff Bound

Let $X_1, X_2, \dots, X_n$ be independent RVs s.t. $X_i = \begin{cases} 1 & \text{w/ pr } p_i \\ 0 & \text{o/w} \end{cases}$

Let $X := \sum_i X_i$, $\mu := \mathbb{E}[X]$

Then $\forall \; \delta \geq 0$

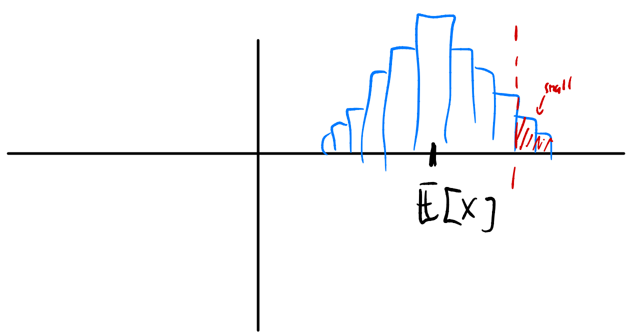
$$\Pr\left(X \geq (1+\delta)\cdot\mu\right) \leq \exp\left(-\mu\delta^2/(2+\delta)\right)$$

And $\forall \; \delta \in (0,1)$

$$\Pr\left(X \leq (1-\delta)\cdot\mu\right) \leq \exp\left(-\delta^2\mu/2\right)$$

→ If $\mu\cdot\delta \approx c\cdot\log n$, get at most $\approx \frac{1}{n^c}$ → "with high probability"

→ Assumes structured RV but exponentially-good concentration



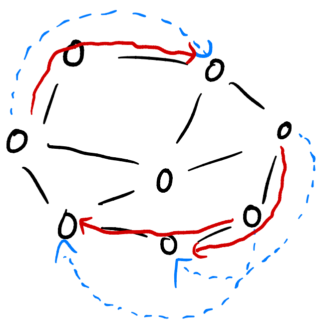
High Level Goal: Sparse network in which easy to route information

Given graph $G=(V,E)$, a <u>demand</u> is a function $D: V \times V \to \{0,1\}$

A demand is <u>unit</u> iff $\sum_v D(u,v), \sum_v D(v,u) \le 1 \quad \forall u \in V$

A set of (directed) paths $\mathcal{P}$ <u>routes</u> D if $D(u,v)=1 \to \exists$ a $u \to v$ path in $\mathcal{P}$

The <u>congestion</u> of e is $\mathrm{con}_{\mathcal{P}}(e) = |\{P \in \mathcal{P} : e \in P\}|$ and $\mathcal{P}$ is $\mathrm{con}(\mathcal{P}) := \max_e \mathrm{con}_{\mathcal{P}}(e)$



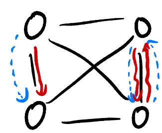
D

$\mathcal{P}$

Congestion $=2$

<u>Observation:</u> Can route any unit demand in K_n w/ congestion ≤ 2

Complete graph on n vertices



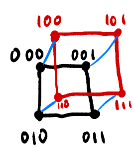
Just send "directly"

Problem: not sparse (many edges)

The <u>n-node hypercube</u> is the graph w/

$V = \{0,1\}^{\log n}$ → n nodes

$\{u,v\} \in E$ iff u,v differ in exactly 1 coordinate



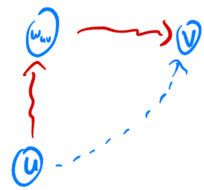
<u>Observation</u> n-node hypercube has $O(n \cdot \log n)$ edges → each node has degree $\log n$

Claim: Can route any unit demand on n-node hypercube w/ congestion $O(\log n)$

Let $\underline{P(u,v)}$ be the path from u to v which "fixes bits" left to right

E.g. $\overset{u}{0101} \to 1101 \to 1001 \to \overset{v}{1011}$

Valiant's Algorithm



$\mathcal{P} = \emptyset$

For each (u,v) s.t. $D(u,v)=1$

Let w_{uv} be a uniformly random node

Add $P(u,w_{uv}) \oplus P(w_{uv},v)$ to $\mathcal{P}$

concatenation

Return $\mathcal{P}$

Clearly $\mathcal{P}$ routes D

Let $\mathcal{P}_1 := \{P(u,w_{uv}) : D(u,v)=1\}$, $\mathcal{P}_2 := \{P(w_{uv},v) : D(u,v)=1\}$

$\mathrm{Con}(\mathcal{P}) \le \mathrm{Con}(\mathcal{P}_1) + \mathrm{Con}(\mathcal{P}_2)$

First bound $\mathrm{Con}(\mathcal{P}_1)$

Fix $e = \{x,y\} \in E$

$$x = (a_1, a_2, \ldots, a_i, \ldots, a_{\log n})$$

$$y = (a_1, a_2, \ldots, \overline{a_i}, \ldots, a_{\log n})$$

Consider $u, \overset{\text{random}}{w_{uv}} : D(u,v)=1$

$$u = (u_1, u_2, \ldots, u_i, \ldots, u_{\log n})$$

$$w_{uv} = (w_1, w_2, \ldots, w_i, \ldots, w_{\log n})$$

$P(u,w_{uv})$ only uses e if

(1) $w_1 = a_1, w_2 = a_2, \ldots, w_{i-1} = a_{i-1}$ and

(2) $u_{i+1} = a_{i+1}, u_{i+2} = a_{i+2}, \ldots, u_{\log n} = a_{\log n}$

(2) is only satisfied for $\le 2^i$-many nodes

(1) is satisfied w/ $\Pr \le \frac{1}{2^{i-1}}$

For each node u w/ $D(u,v)=1$

let $X_u := \mathbb{1}(w_{uv}$ satisfies (1)+(2)$)$ so $\mathbb{E}[X_u] \le \frac{1}{2^{i-1}}$

(a) let $X_e := \sum_u X_u$ so $\mathbb{E}[X_e] \le 2$

(b) Let $\mathcal{B}_e := X_e \ge 2 + 10 \cdot \log n$

so $\overline{\mathcal{B}_e} \to \mathrm{Con}_{\mathcal{P}_1}(e) < 1 + 10 \cdot \log n$

$$\Pr(\mathcal{B}_e) = \Pr(X_e \ge \overset{\mu}{2} \cdot (1 + \overset{\delta}{5 \cdot \log n}))$$

$$\le \exp\left(-\frac{2 \cdot 25 \cdot \log^2 n}{2 + 5 \cdot \log n}\right) \quad \text{(Chernoff)}$$

$$\le \exp(-5 \log n)$$

$$\le n^{-5}$$

(c) So $\Pr\left(\bigcup_e \mathcal{B}_e\right) \overset{\text{(UB)}}{\le} n^2 \cdot n^{-5} \le \frac{1}{n^3}$

So $\Pr(\mathrm{Con}(\mathcal{P}_1) > 2 + 10 \cdot \log n) \le \frac{1}{n^3}$

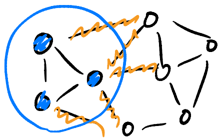
Symmetrically, $\Pr(\mathrm{Con}(\mathcal{P}_2) > 2 + 10 \cdot \log n) \le \frac{1}{n^3}$

So $\Pr(\mathrm{Con}(\mathcal{P}) > 2 + 20 \cdot \log n) \le \frac{2}{n^3}$ (UB)

unpack more here, a bit fast

High Level Goal: Small # edge deletions to disconnect graph

Given $G=(V,E)$, a cut is any non-empty $S \subsetneq V$



S

$\delta(S)$

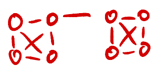
$|\delta(S)| = 5$

The edges of S are $\delta(S) := \{\{u,v\}\in E : u \in S, v \notin S\}$

The size of S is $|\delta(S)|$ and $k := \min |\delta(S)|$

Min-Cut Problem: Given $G=(V,E)$, find a cut $S \subsetneq V$ of size k

Uniformly random cut is a bad alg.



→ 2^n cuts

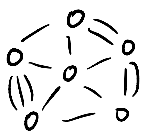
→ But maybe only 1 min cut

Karger's Algorithm

Intuition: Most edges $\notin$ a min cut / uniformly random edge probably not $\in$ fixed min cut

A multiset is a set where elements can appear multiple times

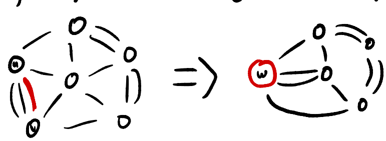
A multigraph is (V,E) where E is a multiset



"parallel" edges

Contracting edge $\{u,v\}$ of (multi)graph $G=(V,E)$ consists of

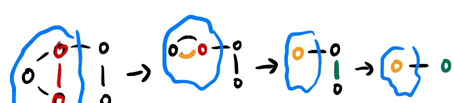
1) Adding a new w to V and $\forall \{u,x\}$ or $\{v,x\}$ add $\{w,x\}$ to E
2) Deleting u,v from V (and their edges)



While $|V|>2$:

Contract a uniformly random $e \in E$

Return cut induced by 1 of remaining vertices

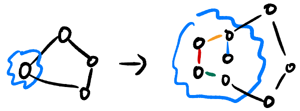


<u>Claim:</u> $\forall S \subseteq V$ if $|\delta(S)| = k$ then $\Pr(S \text{ output}) \geq \frac{1}{n^2}$

A graph w/ $n-i$ nodes where all nodes have deg. $\geq k$ has # edges

$$\frac{1}{2} \cdot \sum_{v \in V} |\delta(v)| \geq \frac{(n-i)k}{2} \text{ edges}$$

Min. deg. always $\geq k$ after i contractions



So after i steps, have $\geq (n-i)k/2$ edges

Let $A_i :=$ event where contract $e \in \delta(S)$ in ith step

$$\Pr(A_1) = \frac{|\delta(S)|}{|E|} = \frac{k}{|E|} \leq \frac{k}{kn/2} = \frac{2}{n}$$

$$\Pr(A_2 | \bar{A}_1) \leq \frac{|\delta(S)|}{(n-1)k/2} = \frac{2}{n-1}$$

...

$$\Pr(A_i | \bar{A}_1 \cap \bar{A}_2 \cap \dots) \leq \frac{|\delta(S)|}{(n-i)\cdot k/2} = \frac{2}{n-i+1}$$

$$\Pr(S \text{ output}) = \Pr\left(\bigcap_{i=1}^{n-2} \bar{A}_i\right) = \Pr(\bar{A}_1) \cdot \Pr(\bar{A}_2 | \bar{A}_1) \cdot \Pr(\bar{A}_3 | \bar{A}_1 \cap \bar{A}_2) \dots \Pr(\bar{A}_{n-2} | \bar{A}_1 \cap \dots \cap \bar{A}_{n-3})$$

$$\geq \left(1 - \frac{2}{n}\right) \cdot \left(1 - \frac{2}{n-1}\right)\left(1 - \frac{2}{n-2}\right) \dots \left(1 - \frac{2}{3}\right)$$

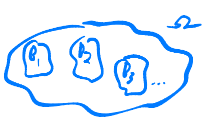
$$= \frac{n-\cancel{2}}{n} \; \frac{\cancel{n-3}}{n-1} \; \frac{n-\cancel{4}}{\cancel{n-2}} \dots \frac{1}{\cancel{3}}$$

$$\geq \frac{1}{n^2}$$

<u>Corollary</u>: # Min cuts $\leq \frac{1}{n^2} << 2^n$

Let $B_i :=$ min cut S_i returned; Suppose ℓ min cuts

All B_i disjoint $\ell/n^2 \leq \sum_{i=1}^{\ell} \Pr(B_i) \leq 1$



Success Probability Boosting

Repeat alg. $\underset{\parallel}{\overset{n^2 \cdot \log n}{}}$ r times, return min cut result

Let F_i = ith computed cut not min

$$
\begin{aligned}
Pr(\text{don't return min cut}) &= Pr(F_1 \cap F_2 \cap \ldots \cap F_r) \\
&= \prod_i Pr(F_i) \\
&\leq \left(1 - \frac{1}{n^2}\right)^r \\
&\leq \exp(-\log n) \\
&= \frac{1}{n}
\end{aligned}
$$