Today

- Basic Tools
- Randomization
- Polyhedral Methods
- Metrics
- Graph Sparsification
- AMA

Basic Tools

Goal: approximate complex functions w/ small # of Xs, $\div$s of simple functions

Powers $\longrightarrow$

exponents $\downarrow$

$$\log n \to \log^2 n \to \log^3 n \dots$$

$$n \to n^2 \to n^3 \dots$$

$$2^n \to 2^{2n} \to 2^{3n} \dots$$

$f = O(g) \leftrightarrow f \le g$

$f = \Omega(g) \leftrightarrow f \ge g$

$f = o(g) \leftrightarrow f < g$

$f = \omega(g) \leftrightarrow f > g$

$f = \Theta(g) \leftrightarrow f = g$

Tricks

$1+x \approx e^x$
- $\nearrow 1+x \le e^x \quad \forall x$
- $\searrow 1+x+\Theta(x^2) \ge e^x \quad \forall x$

$n! \approx \frac{n^n}{e^n}$
- $\nearrow n! \le O\left(\frac{n^n \cdot n}{e^n}\right)$
- $\searrow n! \ge \frac{n^n}{e^n}$

$\binom{n}{k} \approx \left(\frac{ne}{k}\right)^k$
- $\nearrow \binom{n}{k} \le \left(\frac{ne}{k}\right)^k$
- $\searrow \binom{n}{k} \ge \left(\frac{n}{k}\right)^k$

Other: factor, take logs, balance terms

4-Series

$\sum_{i=1}^{n} i = \Theta(n^2)$ $\qquad$ $\sum_{i=1}^{n} \frac{1}{i} = \Theta(\log n)$

$\sum_{i=1}^{n} \frac{1}{i^2} = \Theta(1)$ $\qquad$ $\sum_{i=0}^{\infty} r^i = \Theta(1)$ (for $r \in (0,1)$)

3 Big Inequalities

C-S: $\langle u, v\rangle \le \sqrt{\langle u,u\rangle \cdot \langle v,v\rangle} \quad \forall u, v \in \mathbb{R}^n$

AM-GM: $\left(\prod_{i=1}^{k} x_i\right)^{1/k} \le \frac{\sum_{i=1}^{k} x_i}{k} \quad \forall x \in \mathbb{R}^k$

Jensen's: $f\left(\sum_i p_i x_i\right) \le \sum_i p_i \cdot f(x_i)$

$\forall x \in \mathbb{R}^n$, P a prob. dist.

f convex

$\hookrightarrow$ Jensen for $n=2$

Common Strategies: Guessing, Charging, *Doubling/Halving, Averaging

$\hookrightarrow$ If $x \ge 1$, $x \le n$, x non-decreasing then # times x can double $\le \log n$

Randomization

Gaussians

$Z \sim N(0,1)$ a Standard Gaussian RV $\left(\varphi(x) \approx e^{-x^2}\right)$

$Y = \mu + \sigma Z \sim N(\mu, \sigma^2)$ a non-Standard Gaussian RV $\left(Z \sim N(0,1)\right)$

Rotational Symmetry

Fact: $X = (z_1, z_2, \dots, z_n)$ in a uniformly random direction for $z_i \sim N(0,1)$ independently

Fact: If $X \sim N(0,a^2)$, $Y \sim N(0,b^2)$ then $X+Y \sim N(0, a^2+b^2)$ → Σ_i of Gaussian is a Gaussian

Concentration Framework

a) Show (*) true if all RVs near $\mathbb{E}$

b) Concentration: each RV at $\mathbb{E}$ ($\pm \log n$) whp

c) Union Bound: all RVs " "

Usually via Chernoff Bounds:

$X_1, X_2, \dots, X_n \in \{0,1\}$ ind. $X = \sum_i X_i$, $\mu := \mathbb{E}[X]$

$\Pr(X \geq (1+\delta)\mu) \leq \exp(-\delta^2 \cdot \mu / (2+\delta))$ $\forall \delta > 0$

Alternative: LLL

Probabilistic Method: to show (*) possible, make random choices + show $\Pr(*) > 0$

↳ Saw hypercube routing eg.

Polyhedral Methods

Let $K := \{x : Ax \leq b\}$

↳ a "polyhedron"

LP Feasibility: $\exists\, x \in K$

↑ Poly-time

LP Search: Find $x \in K$

↑ Poly-time

LP Optimization: Find $x \in K$ maxing $\langle c, x \rangle$

Solving LPs

$\approx m^{2^n}$ via Fourier-Motzkin

$\approx m^n$ via enumerating all BFS

Poly-time via ellipsoid

Basic Feasible Solutions

$x \in K$ is a BFS

iff

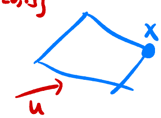
$\text{rank}(\text{tight}(x)) = n$] → $\text{tight}(x) := \{ r_i \in \text{rows}(A) : \langle r_i, x \rangle = b_i \}$; $\text{rank}(V) :=$ max # LI vectors $\subseteq V$

iff

x is an extreme point] → $\nexists\, y, z \in K$ s.t. $x \in \underbrace{[y, z]}_{\{p \cdot y + (1-p) \cdot z \,:\, p \in [0,1]\}}$

iff

x is a vertex] → $\exists\, u$ s.t. $\langle x, u \rangle > \langle x', u \rangle \ \forall x' \neq x \in K$



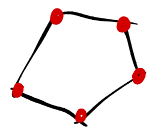
<u>Fact:</u> If $K = \left\{ x : \begin{matrix} Ax \leq b \\ x \geq 0 \end{matrix} \right\} \neq \emptyset$ then K has a BFS

Also, if $\text{OPT} := \max_{x \in K} \langle c, x \rangle < \infty$ then $\exists$ BFS x s.t. $\langle c, x \rangle = \text{OPT}$

Geometric View

<u>Fact:</u> K is the intersection of m halfspaces + convex → $[x, y] \subseteq K \ \forall x, y \in K$

<u>Fact:</u> If K is bounded (i.e. a polytope) w/ BFSs V then $K = \text{Con}(V)$



Two Uses

1) Integrality: all BFSs $\in \mathbb{Z}^n$] → Find an optimal BFS to solve problem

2) Randomized Rounding: $x \in K \overset{\text{Round}}{\rightsquigarrow} x \in K \cap \mathbb{Z}^n$] → Start w/ optimal x, compare to OPT_{LP}

Metrics

Metrics mathematically formalize distance

LDDs: given (V,d) $\Delta > 0$, $\exists$ distribution over Δ-diameter clusterings s.t.

$$\Pr(u,v \text{ separated}) \le O(\log n) \cdot \frac{d(u,v)}{\Delta}$$

↳ Used to round multicut LP

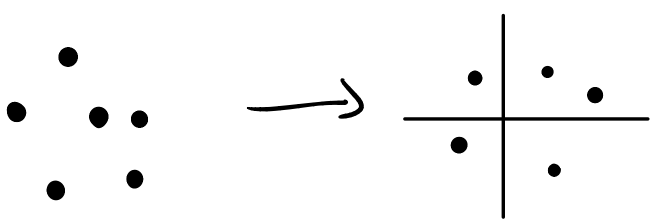
An **embedding** of (V,d) into (V',d') is a function $f: V \to V'$

f has **distortion α** if $d(u,v) \le d'(f(u),f(v)) \le \alpha \cdot d(u,v)$ $\forall u,v \in V$

$$\begin{tikzcd}
 & (V,\delta) \arrow[dl, "\text{Bourgain: } \alpha = O(\log n)"'] \arrow[dr] \arrow[ddr, bend left] & \\
(V \subseteq \mathbb{R}^n, d) \arrow[d, "\text{JL: } \alpha = 1+\epsilon"] & & (V \subseteq \mathbb{R}^n, d_1) \arrow[d, phantom, "\equiv"] \\
(V \subseteq \mathbb{R}^{O(\log n/\epsilon^2)}, d) & & \mathrm{Cone}(\mathrm{Cut}(V))
\end{tikzcd}$$

Euc. distance

→ Gave $O(\log n)$-apx for Sparsest Cut problem



Graph Sparsification

"Every graph is (approximately) a tree"

Given graph $G=(V,E,w)$

Cuts: $\exists$ tree $T=(V,E_T,w_T)$ s.t. $\forall s,t \in V$ if $W \subseteq V$ is an s-t MC in T then

1) W is an s-t MC in G
2) $w(\delta_G(W)) = w_T(\delta_T(W))$

Distances: $\exists$ a distribution over trees Υ s.t. $\forall u,v \in V$

1) $d(u,v) \le d_T(u,v)$ $\forall u,v \in V,\ T \in \Upsilon$
2) $\mathbb{E}_{T \sim \Upsilon}[d_T(u,v)] \le O(\log n) \cdot d(u,v)$

(Also subtrees but is $\tilde{O}(\log n)$)

Routings: $\exists$ a distribution over trees Υ s.t. obliviously routing via Υ is $\hat{O}(\log n)$-congestion competitive