Today

1) Asymptotic Hierarchy

2) Big O + Friends

3) Tricks for Asymptotic simplification

Goal: approximate complex functions w/ small number of $\times$s, $\div$s of simple functions

Simple Functions of n

Powers

$\log n \rightarrow \log^2 n \rightarrow \log^3 n \ldots$

$n \rightarrow n^2 \rightarrow n^3 \ldots$

exponents

$2^n \rightarrow 2^{2n} \rightarrow 2^{3n} \ldots$

Non-Egs

$10 \cdot 2^n n^2 + n$

$\log^2 n + 10n/\log n$

$2^{10 \log n} = \left(2^{\log n}\right)^{10} = n^{10}$

approximate

Egs

$2^n n^2$

$n/\log n$

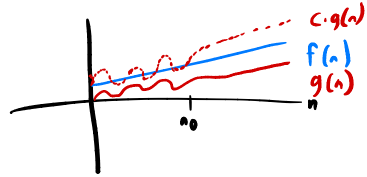
n^{10}

Big O:

$f(n) = O(g(n))$ if $\exists$ c, n_0 s.t.

$|f(n)| \leq c \cdot g(n) \quad \forall n \geq n_0$

$|\cdot|$ almost always forgettable b/c $f(n) \geq 0 \ \forall n$



E.g. RHS = O(LHS)

Strongeness 1 $f(n) = 2^{O(\log n)}$

$$f(n) = g(O(h(n))) \iff \exists\ h'(n) = O(h(n)) \text{ s.t. } f(n) = g(h'(n))$$

Note $O(2^{\log n}) \not\Rightarrow 2^{O(\log n)}$

Strongeness 2 $10\varepsilon^2 = O(\varepsilon)$

$f(\varepsilon) = O(g(\varepsilon))$ if $\exists\ C, \varepsilon_0$ s.t. $\forall\ 0 \le \varepsilon \le \varepsilon_0$ we have

$$|f(\varepsilon)| \le C \cdot g(\varepsilon)$$

Strongeness 3 $n \cdot \varepsilon^2 = O(\varepsilon)$ or $n\varepsilon^2 = O(n)$

→ Disambiguate w/ $n\varepsilon^2 = O_{\varepsilon \to 0}(\varepsilon)$ $n\varepsilon^2 = O_{n \to \infty}(n)$

→ Rest w/ n, drop n from fns.

Inequality Analogy

$$f = O(g) \longleftrightarrow f \le g$$

$$f = \Omega(g) \longleftrightarrow f \ge g$$

$$f = \Theta(g) \longleftrightarrow f = g$$

$$f = o(g) \longleftrightarrow f < g$$

$$f = \omega(g) \longleftrightarrow f > g$$

$f = \Omega(g)$ if $\exists$ c, n_0 s.t. ← Knuth dfn.?

$$|f(n)| \ge c \cdot g(n) \quad \forall\, n \ge n_0$$

E.g. $2^n = \Omega(n)$

$f = \Theta(g)$ if

$$f = O(g) \quad \underline{\text{and}} \quad f = \Omega(g)$$

E.g. $10 \cdot 2^n = \Omega(2^n)$ and $10 \cdot 2^n = O(2^n)$ so $10 \cdot 2^n = \Theta(2^n)$

How to separate 2 cases above?

$f = o(g)$ if $\lim\limits_{n\to\infty} \dfrac{f(n)}{g(n)} \to 0$

$f = \omega(g)$ if $\lim\limits_{n\to\infty} \dfrac{f(n)}{g(n)} \to \infty$

E.g. $n = o(2^n)$

Even Simpler Notation

$f(n)$ is a Polynomial if it is of the form

$$f(n) = a_d n^d + a_{d-1} \cdot n^{d-1} + \ldots + a_0$$

$(a_i \in \mathbb{R} \ \forall i)$

(Write as $f(n) = \text{Poly}(n)$)

$f = \tilde{O}(g)$ if $f = h \cdot O(g)$ for some $h = \text{Poly}(\log(g))$

↳ Often notated $f = h \cdot \text{Poly}(\log g)$

$f = \tilde{\Omega}(g)$ if $f = \Omega(g) / \text{Poly}(\log g)$

E.g. $10 \cdot \log^2 n \cdot n^2 = \tilde{O}(n^2)$

$10 \, n^2 \cdot 2^n = \tilde{O}(2^n)$

Non-Obvious Simplifications

$$\sqrt{1-x} \overset{①}{\leq} e^{-x/2}$$

$$\sqrt{1-\frac{1}{n}} \overset{①}{\leq} \exp(-1/2n) \overset{①}{\leq} 1 - \frac{1}{2n} + \Theta\left(\frac{1}{n^2}\right)$$

$$\ln(n+1) \overset{②}{=} \ln\left(n \cdot \frac{n+1}{n}\right) = \ln\left(n\left(1+\frac{1}{n}\right)\right)$$

$$= \ln n + \ln\left(1+\frac{1}{n}\right)$$

$$\overset{①}{\leq} \ln n + \frac{1}{n}$$

$x = y \cdot \log y$ ③ $\quad \log x = \log y + \log\log y$

$y = f(x)$

$$\log x = \Theta(\log y)$$

$$y \log x = \Theta(y \cdot \log y) = \Theta(x)$$

$$\to y = \Theta(x / \log x)$$

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$$\min_t \left[\frac{n}{\sqrt{t}} + \sqrt{t} \cdot \ln t\right] \quad (*)$$

$$\frac{n}{\sqrt{t}} = \sqrt{t} \cdot \ln t \to n = t \cdot \ln t$$

$$\to t = \theta(n / \ln n)$$

$$\to (*) \leq \theta\left(\sqrt{n \cdot \ln n}\right)$$

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$$n! = n(n-1)(n-2) \ldots 1$$

$$\binom{n}{k} =$$

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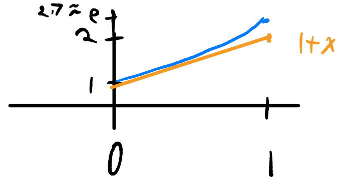
Tricks for Simplifying



1) $e^x \approx 1+x$ (for small x)

$1+x \le e^x \quad (\forall x)$

$x \le -1 \to$ LHS < 0 but $e^x \ge 0$ so ✓

$$x \in (-1,0) \to e^x = 1+x+\underbrace{\left(\frac{x^2}{2}+\frac{x^3}{3!}\right)}_{\ge \frac{x^2+x^3}{3!} \ge 0}+\underbrace{\left(\frac{x^4}{4!}+\frac{x^5}{5!}\right)}_{\ge \frac{x^4+x^5}{5!} \ge 0}+\dots$$

$x \ge 0 \to e^x = 1+x+$ positive stuff $\ge 1+x$

$e^x \le 1+x+\underset{x\to 0}{\Theta}(x^2) \quad (\forall x)$

$$e^x = 1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\dots \le 1+x+x^2\left(\frac{1}{2}+x\frac{1}{4}+x^2\frac{1}{8}+\dots\right)$$

$$\le 1+x+\Theta(x^2)$$

2) Factor into Simplified Form

$$f = \frac{f}{g}\cdot g$$

$\ne$ simple   simple

3) Take Logs

$$f \to \log f$$

4) Balance Terms

$$\min_t \left[f(t)+g(t)\right] \le f(t^*)+g(t^*)$$

t s.t. $f(t^*)=g(t^*)$ ← End of Class

5) Deal w/ Common Troublemakers: $n!$, $\binom{n}{k}$

$\rightarrow n! \approx n^n / e^n$

Lower Bound

$$e^n = \underbrace{1 + n + \frac{n^2}{2!} + \frac{n^3}{3!}}_{\geq 0} + \dots + \frac{n^n}{n!} + \underbrace{\dots}_{\geq 0}$$

So $n! \geq \frac{n^n}{e^n}$

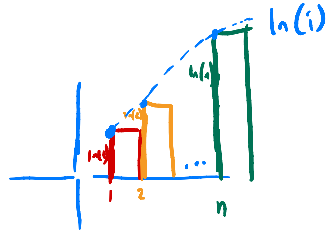
Easy UB

$$n! = n(n-1)(n-2)\dots(1) \leq n \cdot n \cdot n \dots n = n^n$$

Better UB

take logs ③

$$\ln(n!) = \sum_{i=1}^{n} \ln(i) = \text{Area of rectangles} \leq \text{Area under } \ln(i)$$



$$= \int_1^{n+1} \ln(i)\, di$$

$$= i \ln i - i \Big|_1^{n+1}$$

$$= (n+1)\ln(n+1) - n$$

$$\leq (n+1)\left(\ln n + \frac{1}{n}\right) - n$$

$$= n \ln n + \ln n - n + O(1) \quad (*)$$

untake Logs

$$n! \leq e^{(*)} = O\left(\frac{n^n \cdot n}{e^n}\right)$$

$$\boxed{\text{Stirlings Formula: } n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n}$$

$\rightarrow \binom{n}{k} \approx \left(\frac{ne}{k}\right)^k$

UB: $$\binom{n}{k} := \frac{n!}{(n-k)!\,k!} = \frac{(n)(n-1)(n-2)\dots(n-k+1)}{k!}$$

$$\leq \frac{n^k}{k!}$$

$$\leq \left(\frac{ne}{k}\right)^k$$ (by factorial bound)

LB: $$\binom{n}{k} = \frac{(n)(n-1)(n-2)\dots(n-k+1)}{k(k-1)(k-2)\dots(1)}$$ but $\frac{n-i}{k-i} \geq \frac{n}{k}$

$$\geq \left(\frac{n}{k}\right)^k$$

b/c $n(k-i) \leq k(n-i)$

$-n \leq -k$

$n \geq k$