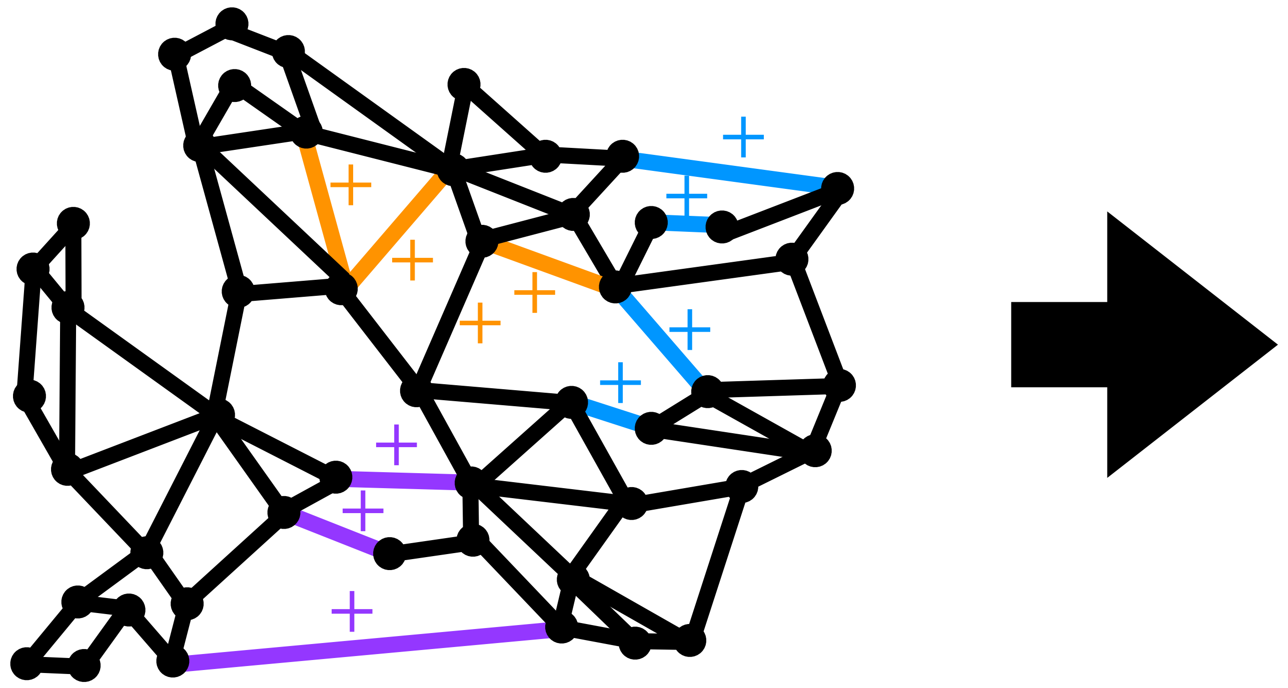
Simple Length-Constrained Expander Decompositions

(SOSA 2026)



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D Ellis Hershkowitz
Brown

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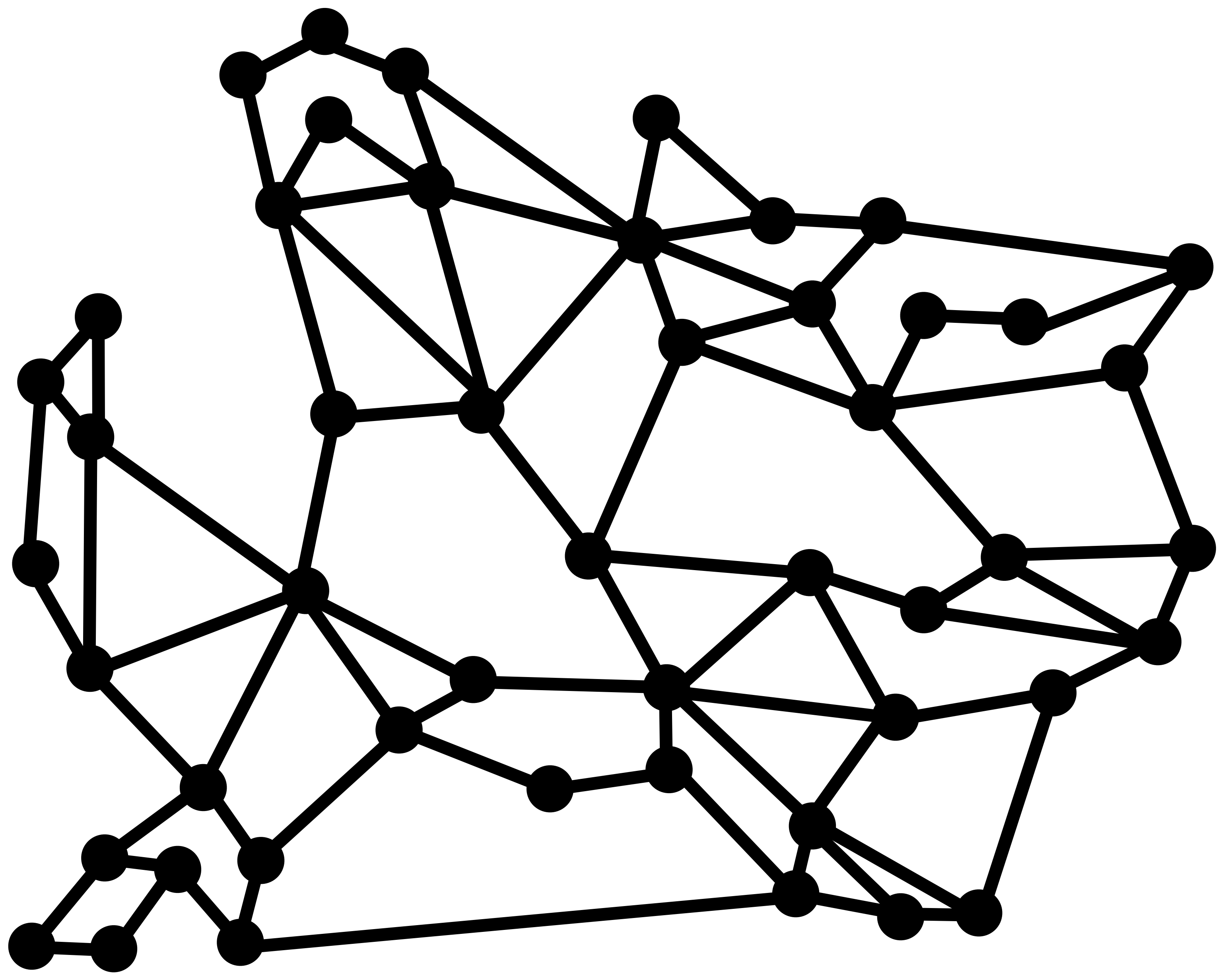
U. Michigan

Zihan Tan
U. Minnesota

How to Solve Your Favorite Graph Problem

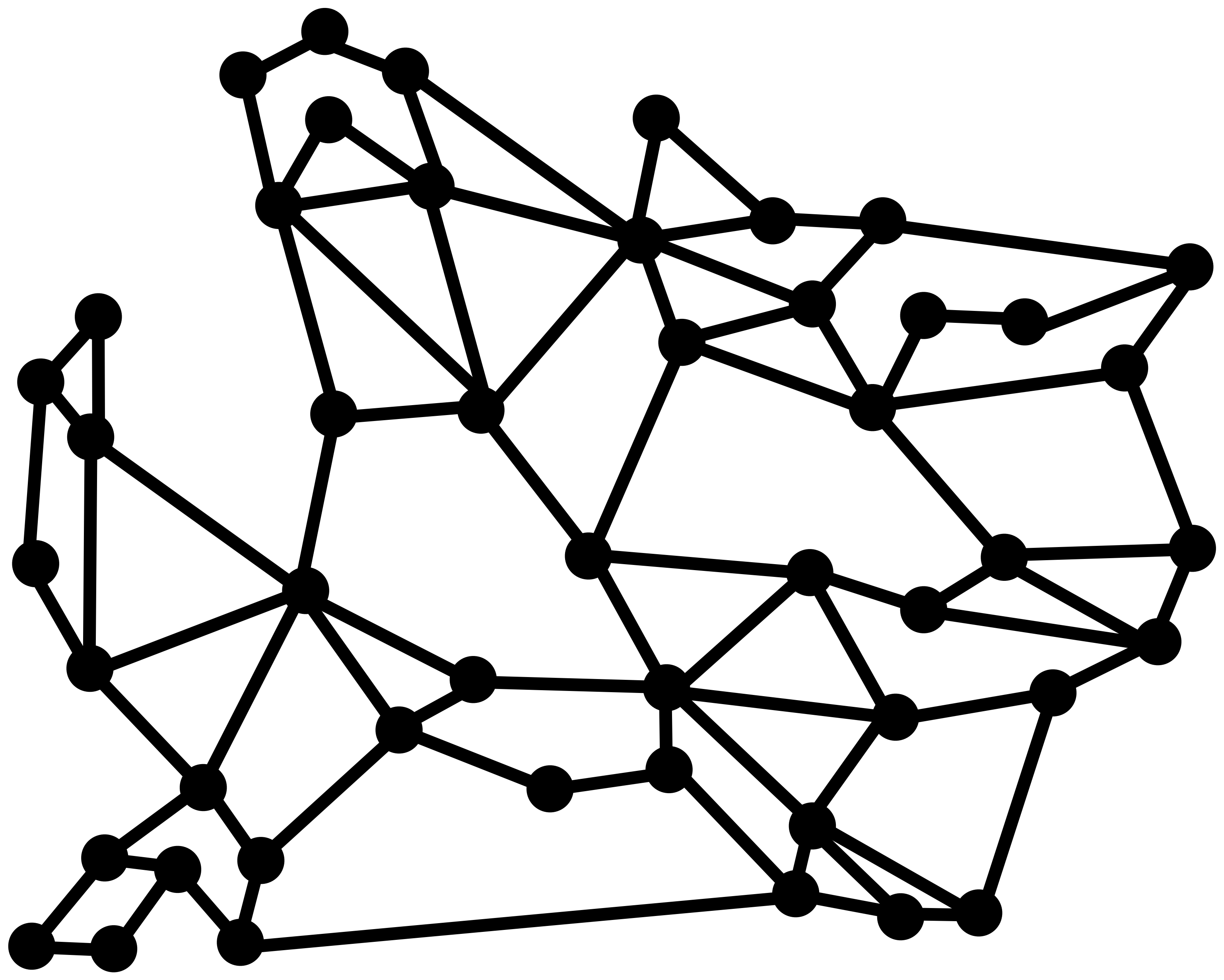
Graph Decomposition Approach



How to Solve Your Favorite Graph Problem

Graph Decomposition Approach

1. **Graph Decomposition**
   add to graph "modifications"
   to make it "nice"



How to Solve Your Favorite Graph Problem

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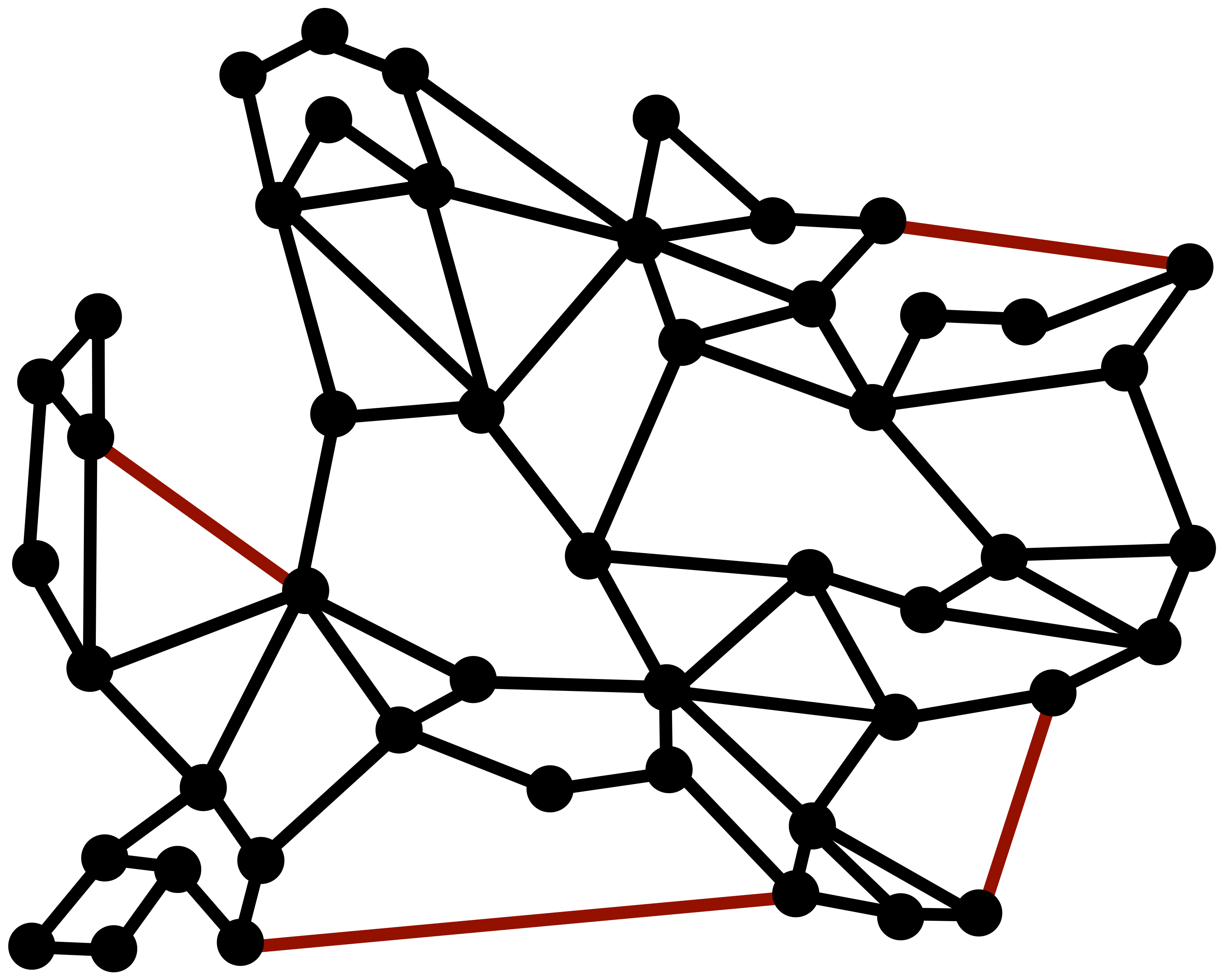

How to Solve Your Favorite Graph Problem

Graph Decomposition Approach

1. **Graph Decomposition**
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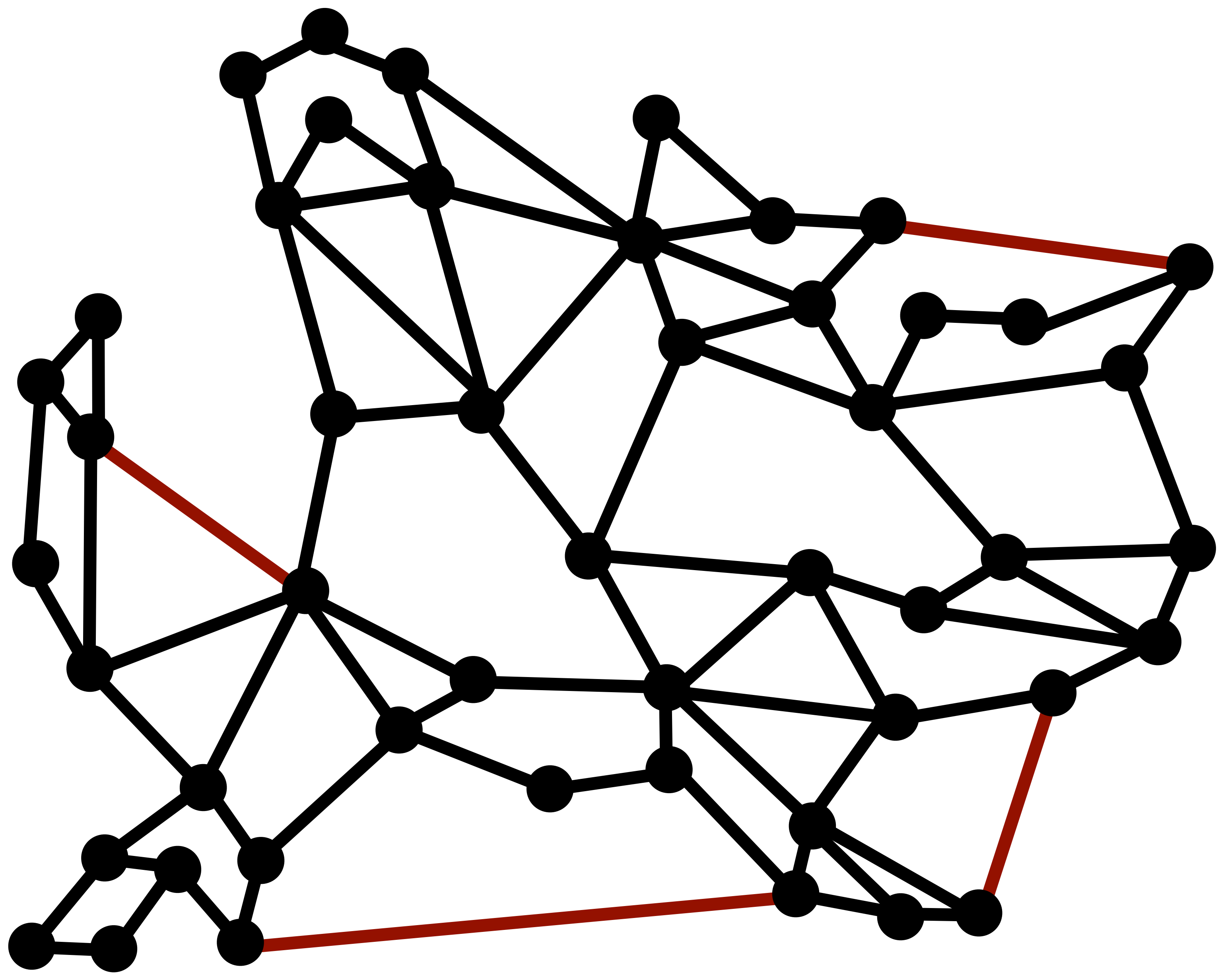
2. **Solve Problem**
   solve problem on nice graph



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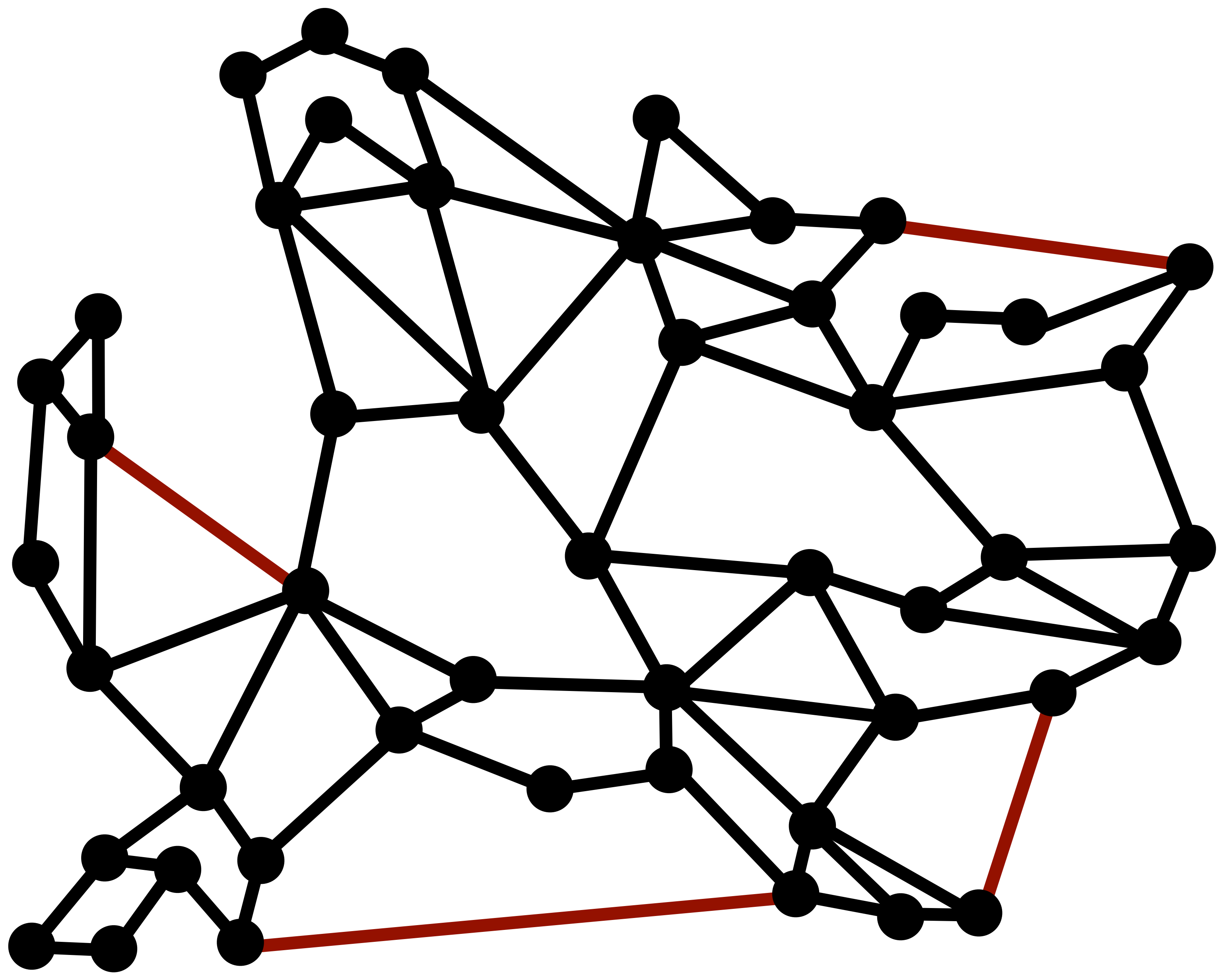
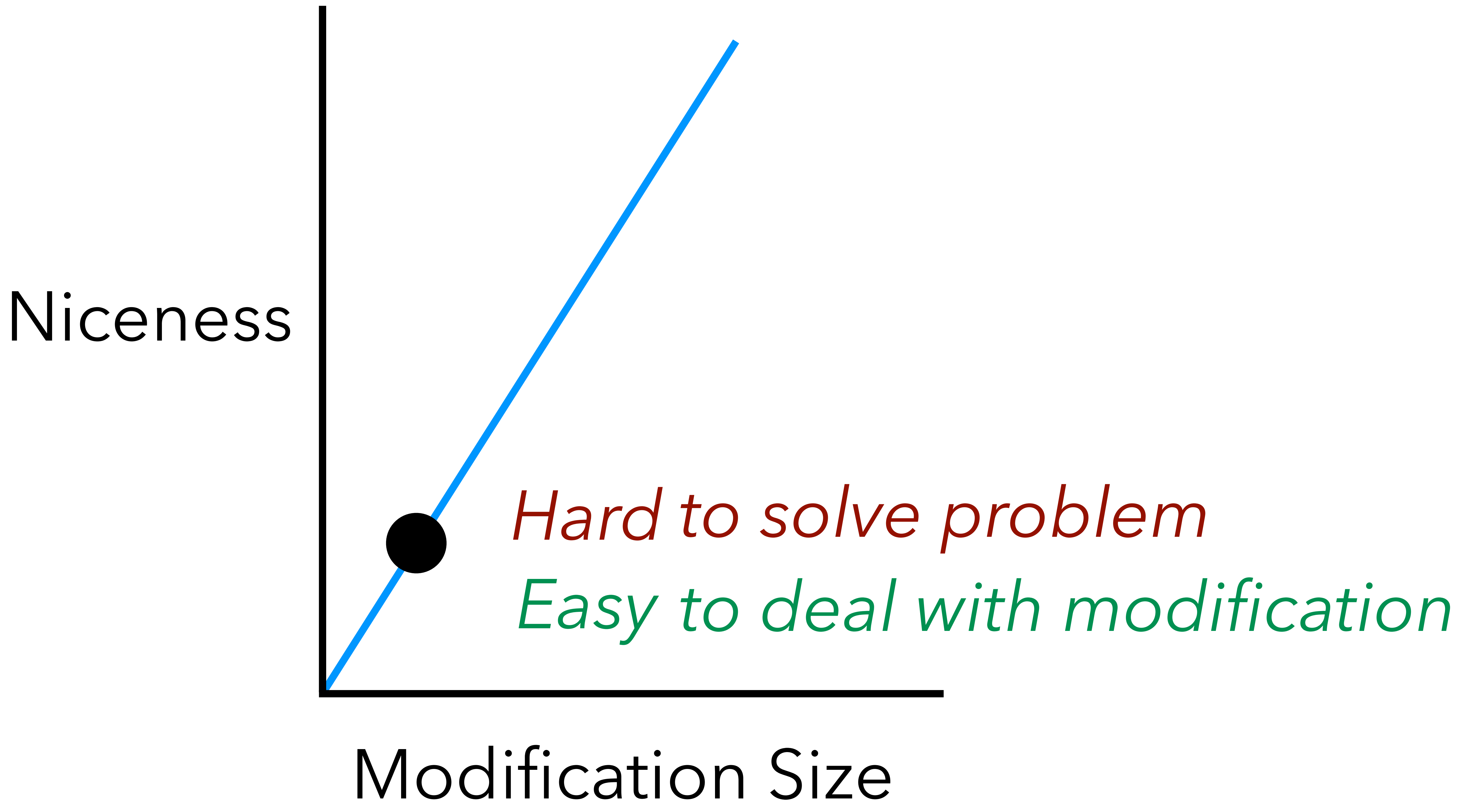
3. **Clean Up**
   deal with modifications

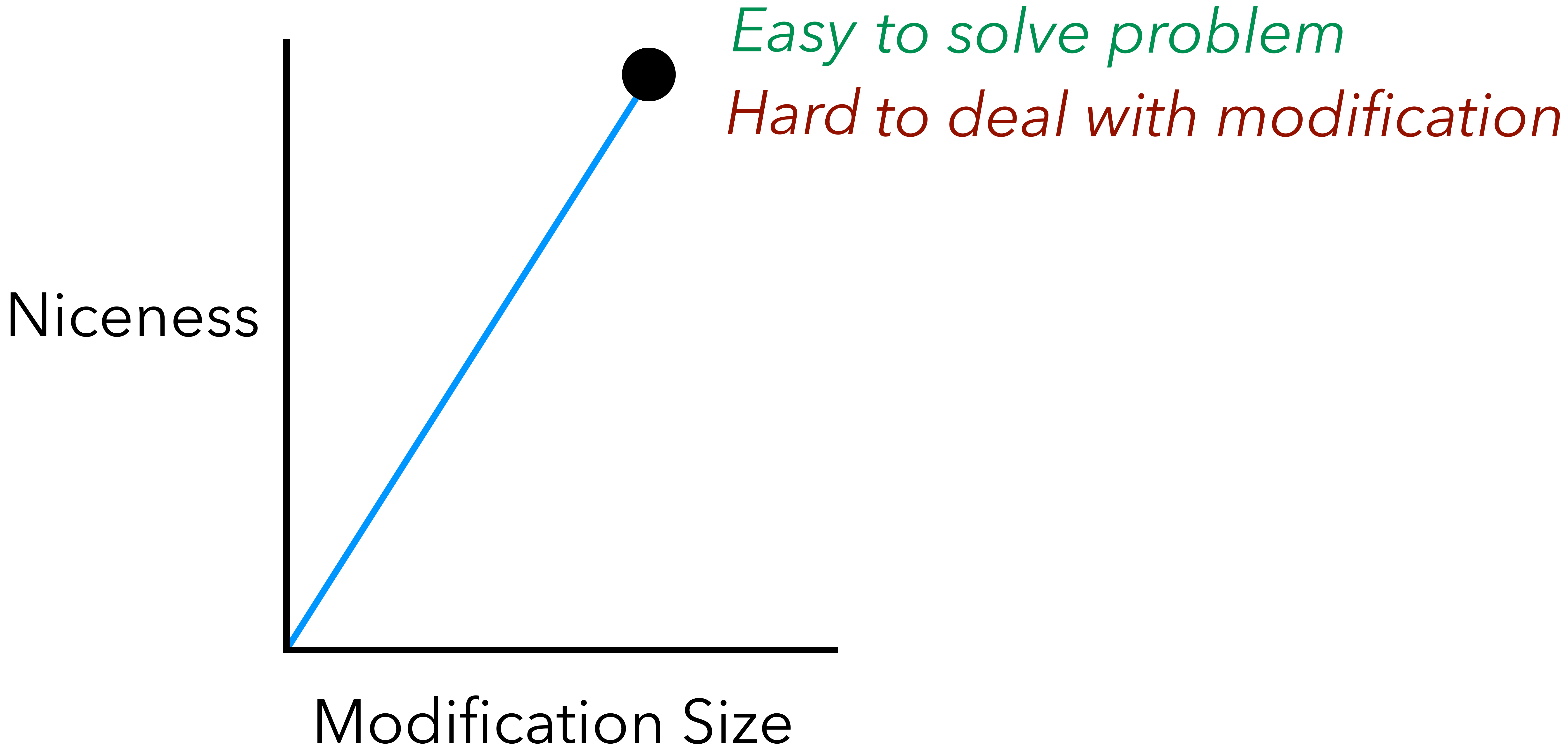


How to Solve Your Favorite Graph Problem



Graph Decomposition Size-Niceness Tradeoff

How to Solve Your Favorite Graph Problem



Graph Decomposition Size-Niceness Tradeoff

Length-Constrained Expander Decompositions

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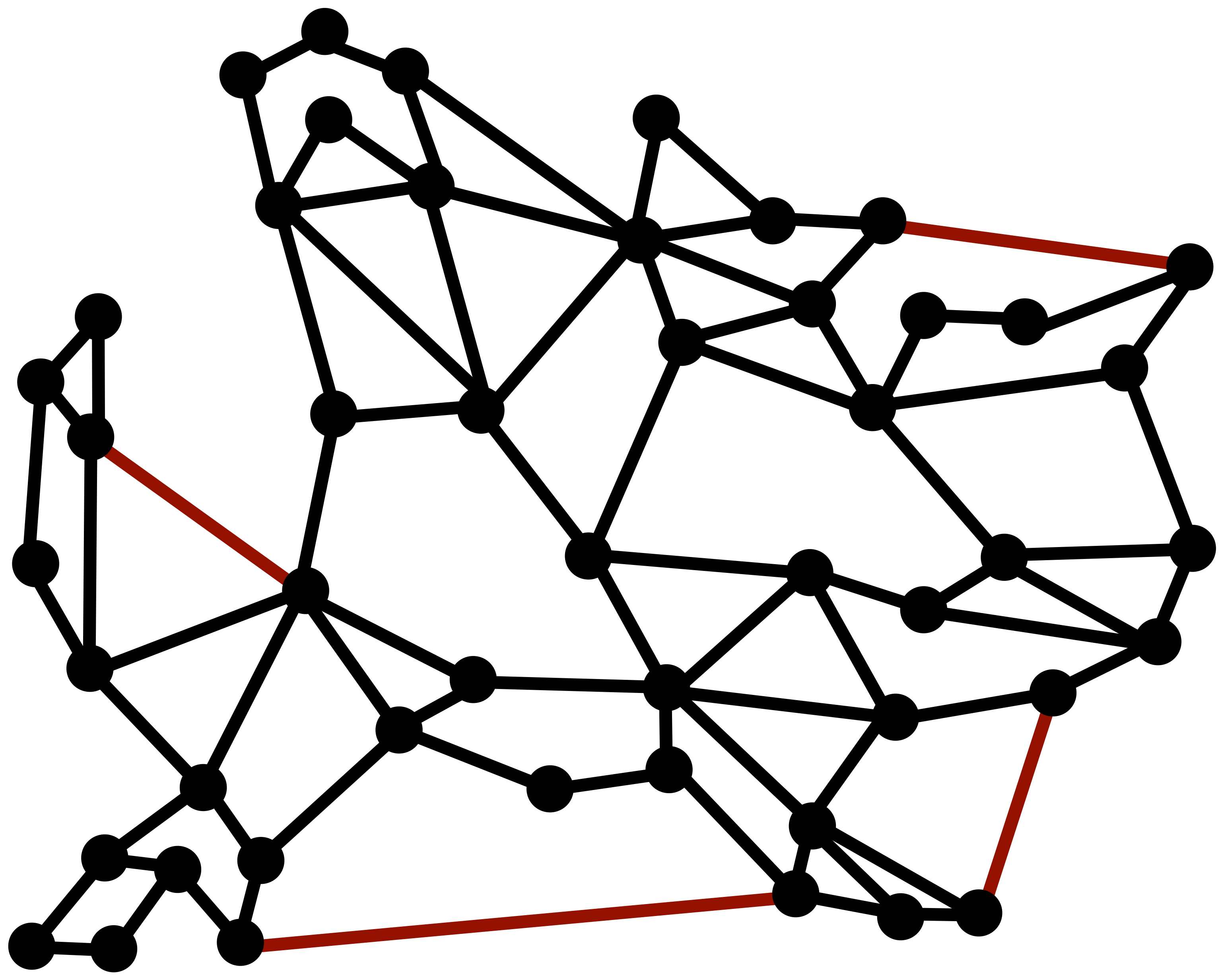

Length-Constrained Expander Decompositions

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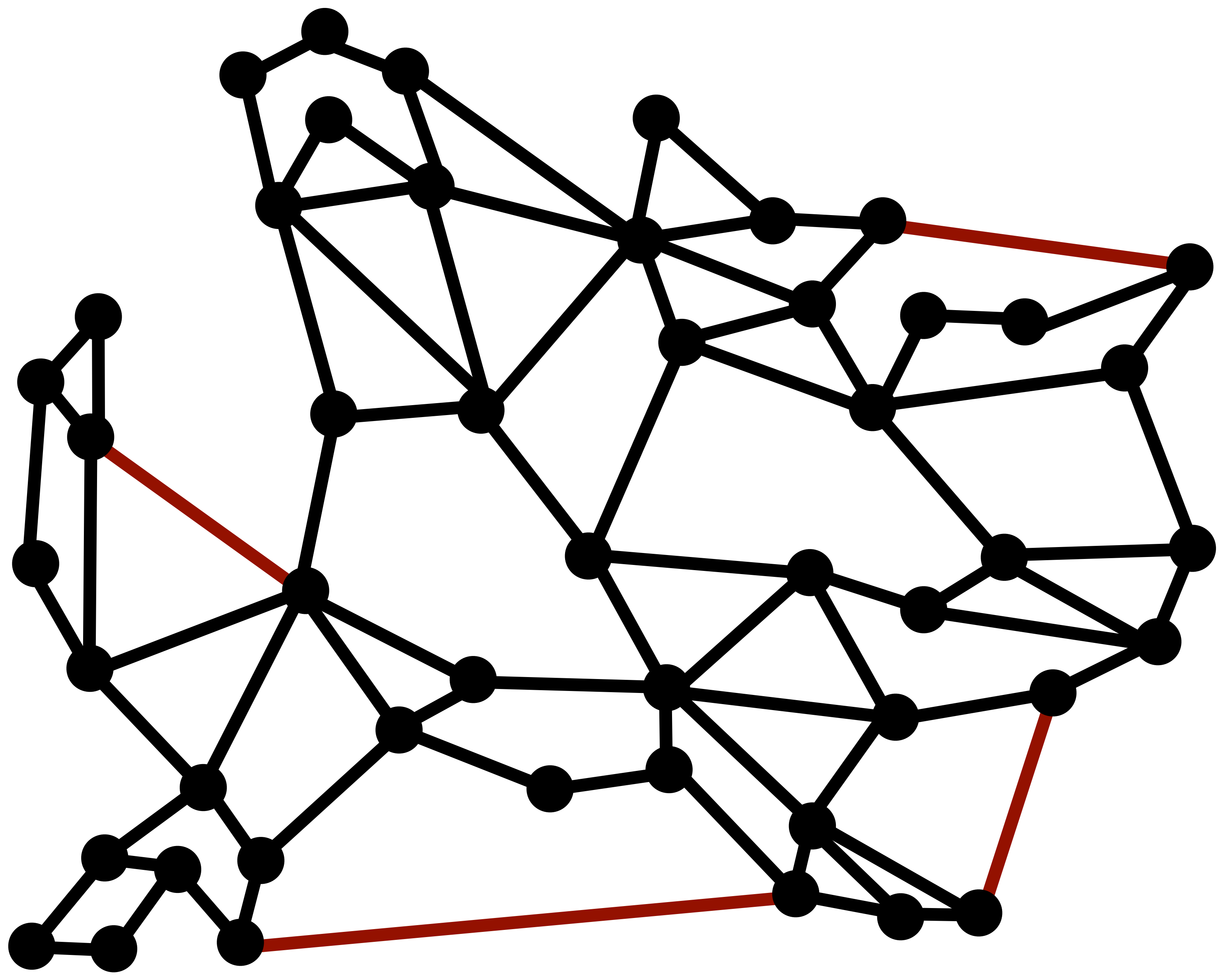
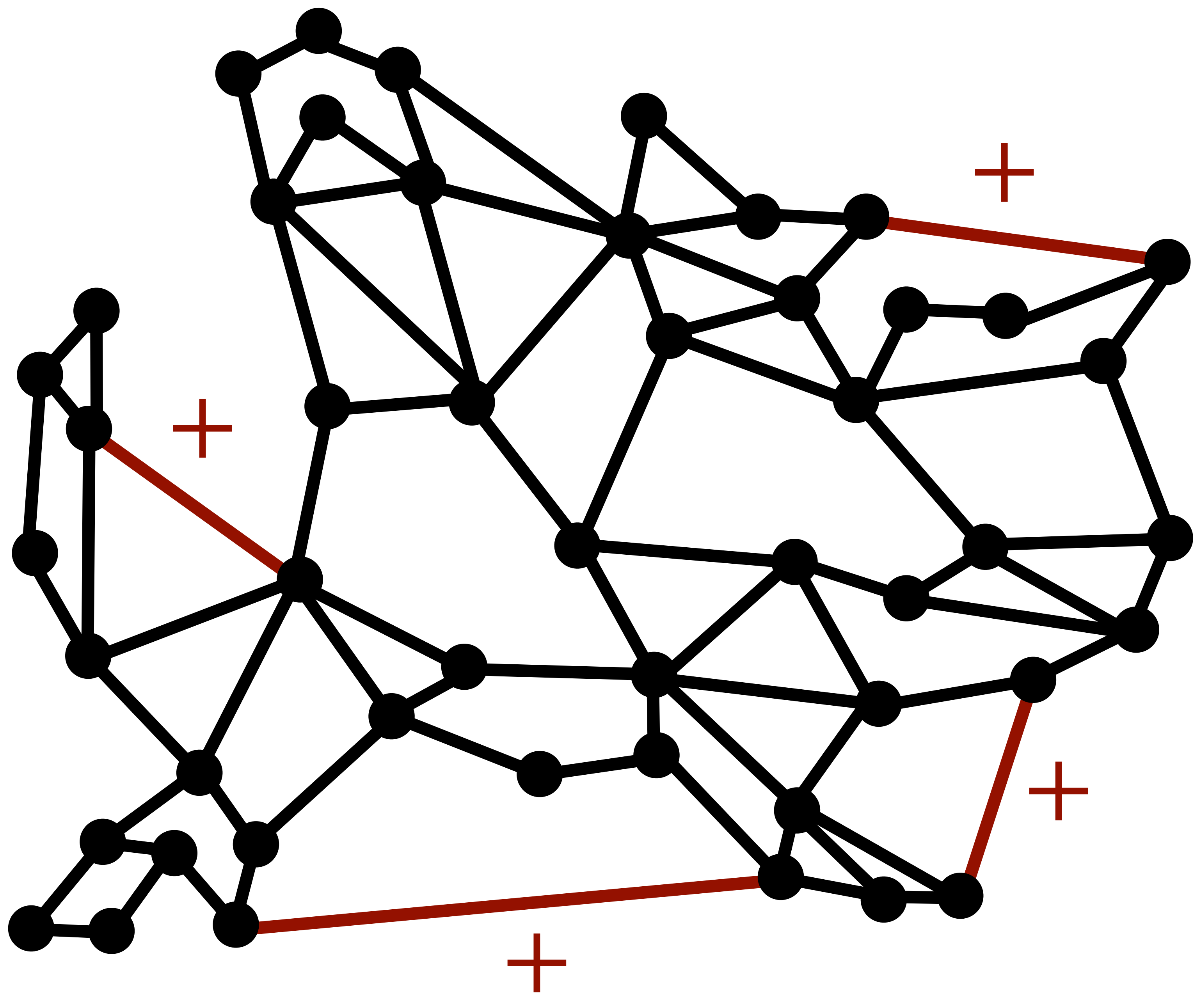


Length-Constrained Expander Decompositions

Graph Decomposition Approach

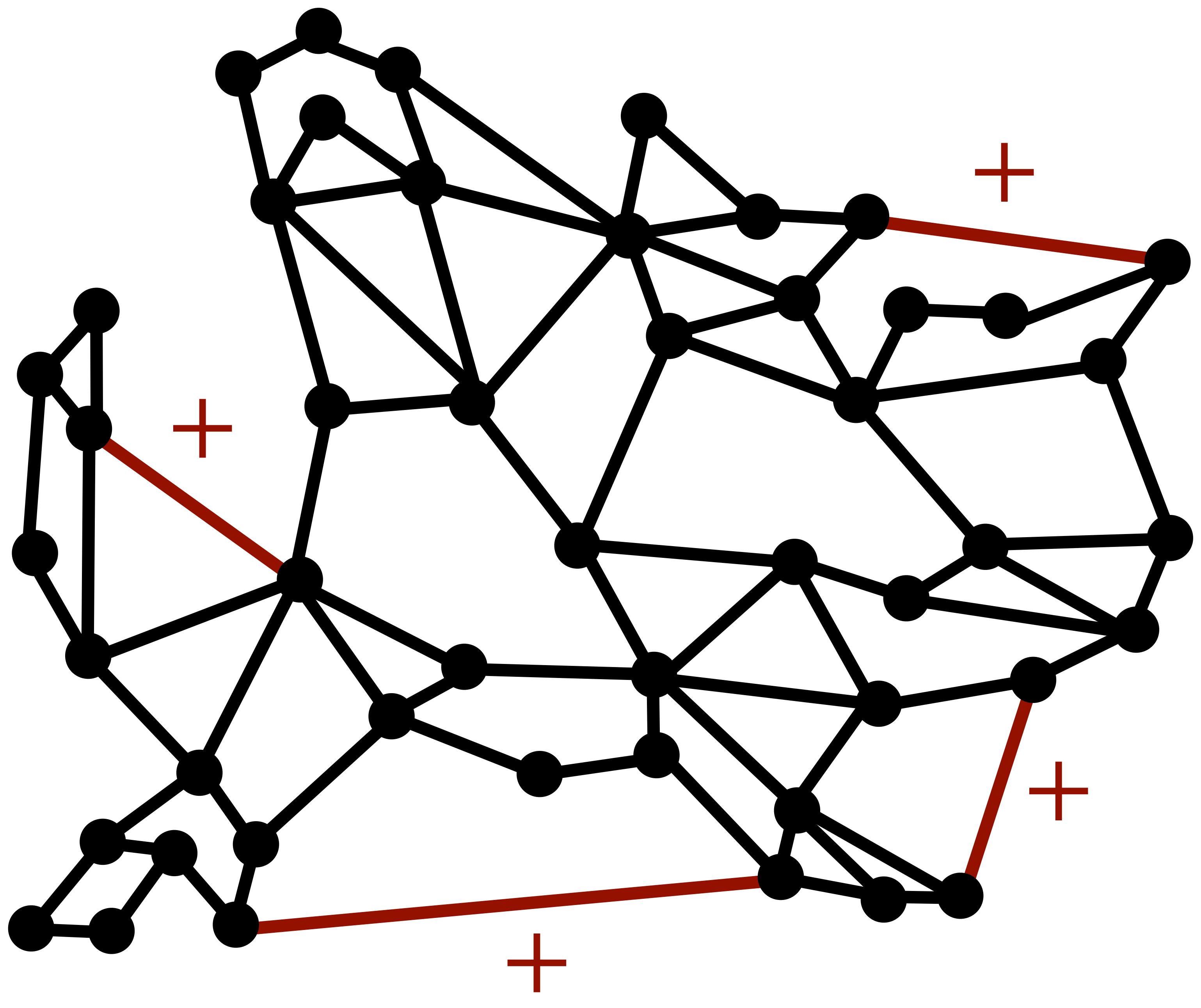
1. **Length-Constrained Expander Decomposition**
   add to graph edge length increases
   to make it "nice"
2. **Solve Problem**
   solve problem on nice graph
3. **Clean Up**
   deal with modifications



Length-Constrained Expander Decompositions

Graph Decomposition Approach

1. **Length-Constrained Expander Decomposition**
   add to graph edge length increases
   to make it a length-constrained expander
2. **Solve Problem**
   solve problem on nice graph
3. **Clean Up**
   deal with modifications



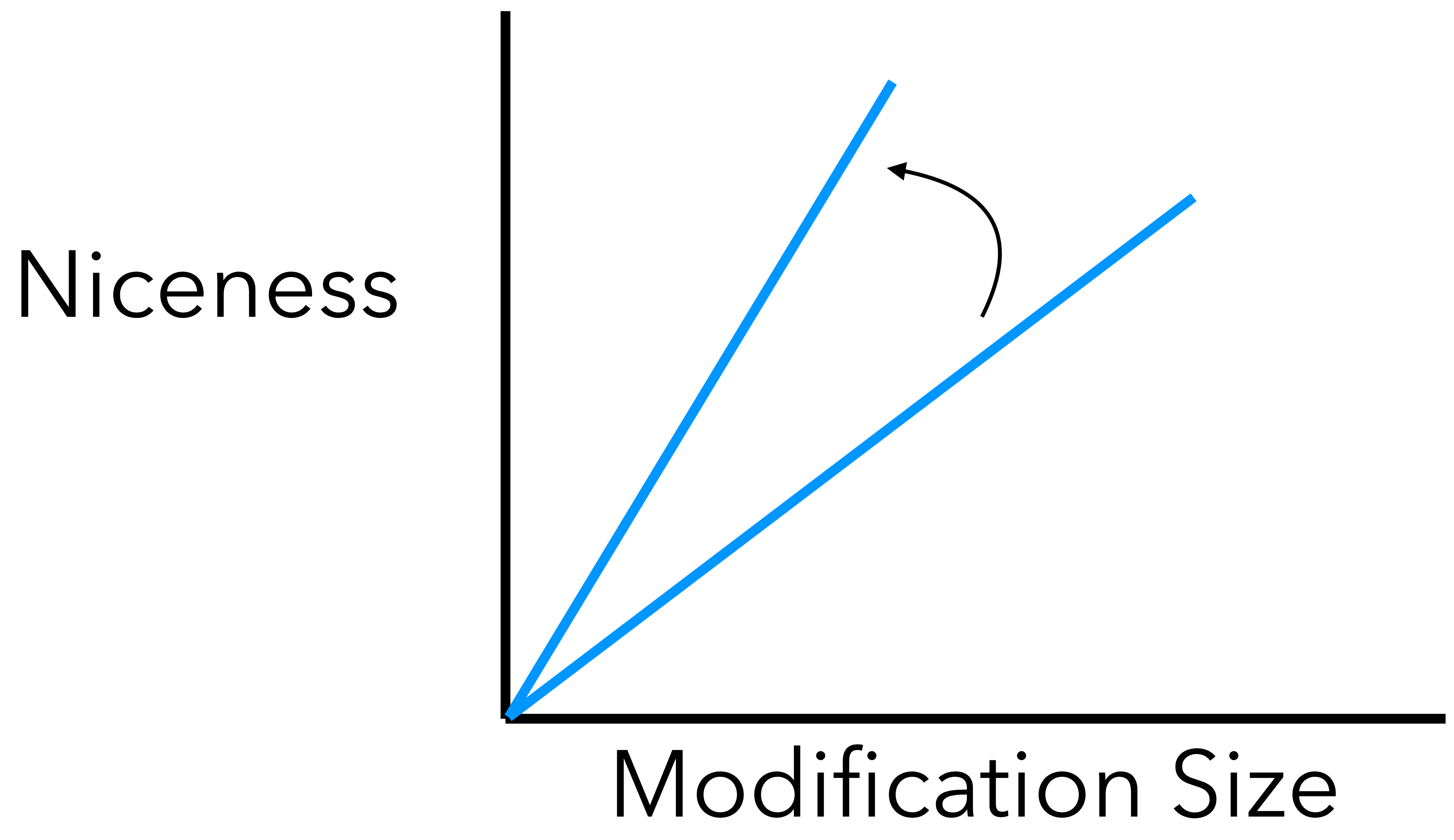
Length-Constrained Expander Decompositions

Graph decomposition approach with LC EDs gives SOTA for:

- Approximate Min-Cost Multi-Commodity Flow [H**H**LRS STOC24]
- Deterministic Distance Oracles [HLS FOCS24]
- $(1+\epsilon)$-Approximate Parallel Min Cost Flow [HJLSW FOCS25]

Our Main Result (Informally)

Theorem [BHHT]. Simple proof of the existence of length-constrained expander decompositions with improved size-niceness tradeoffs



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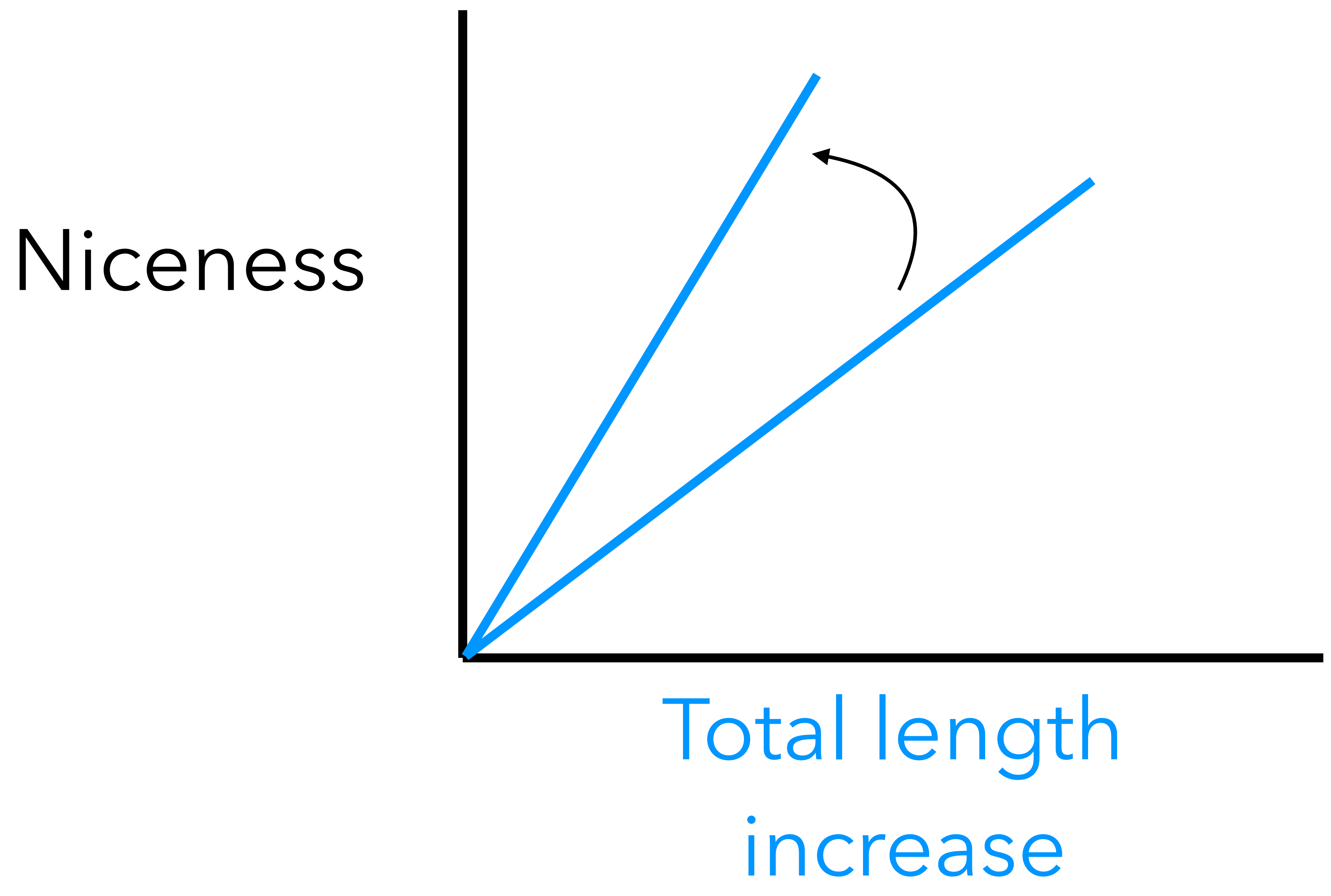

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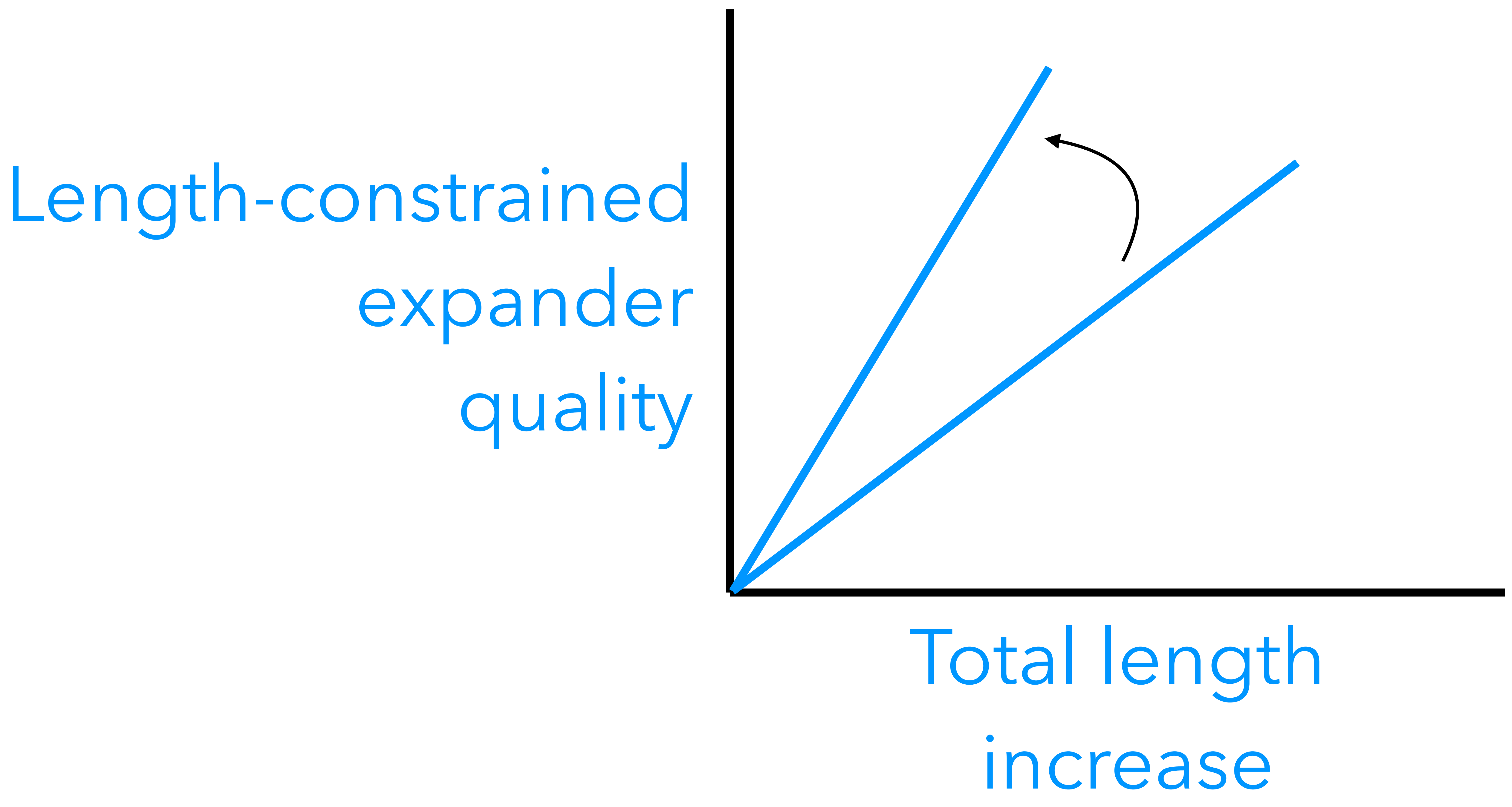
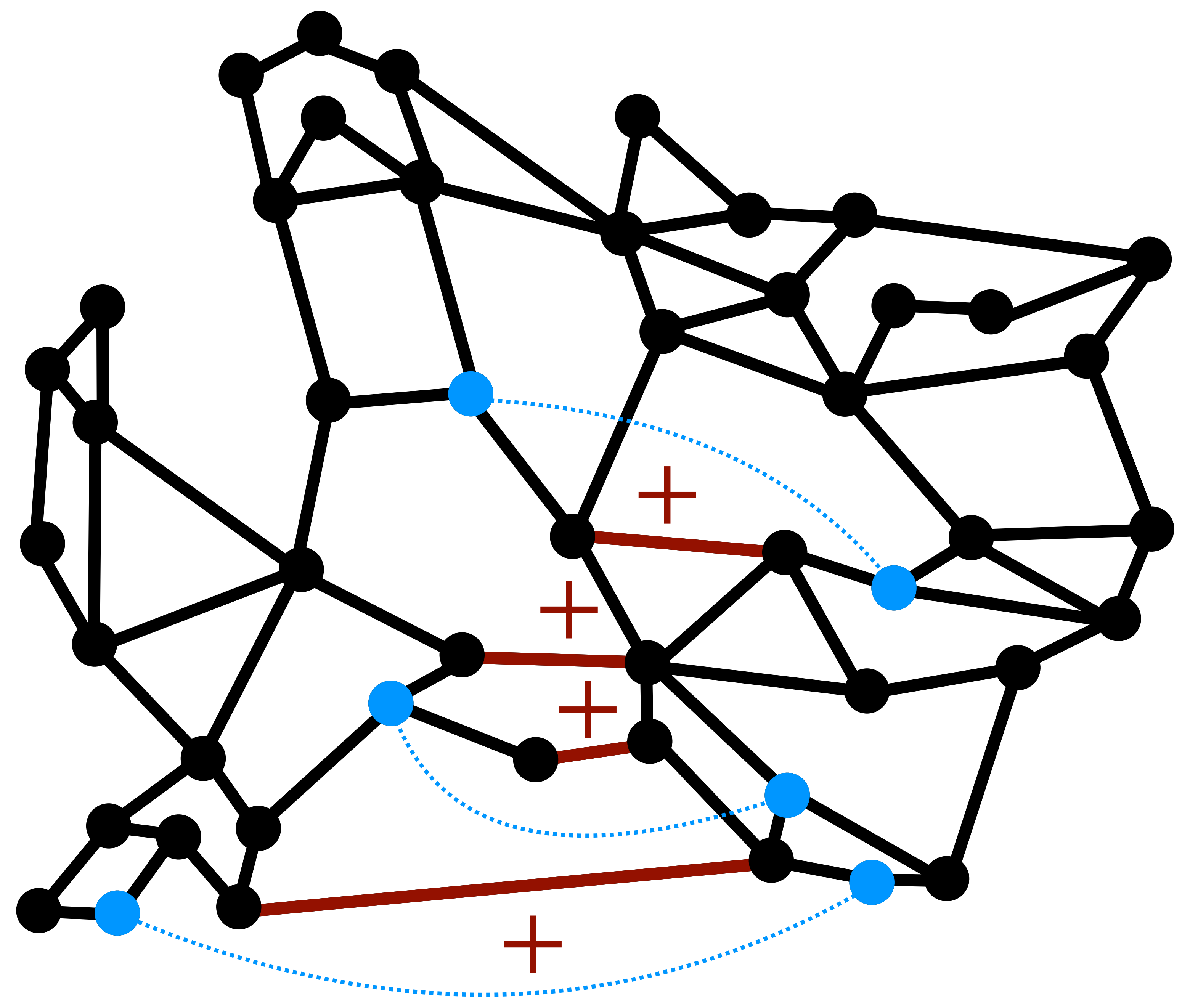


Defining Length-Constrained EDs

Sparse Length-Constrained Cuts

Sparse LC Cut Informally.
small total length increase that makes
many close vertex pairs
far

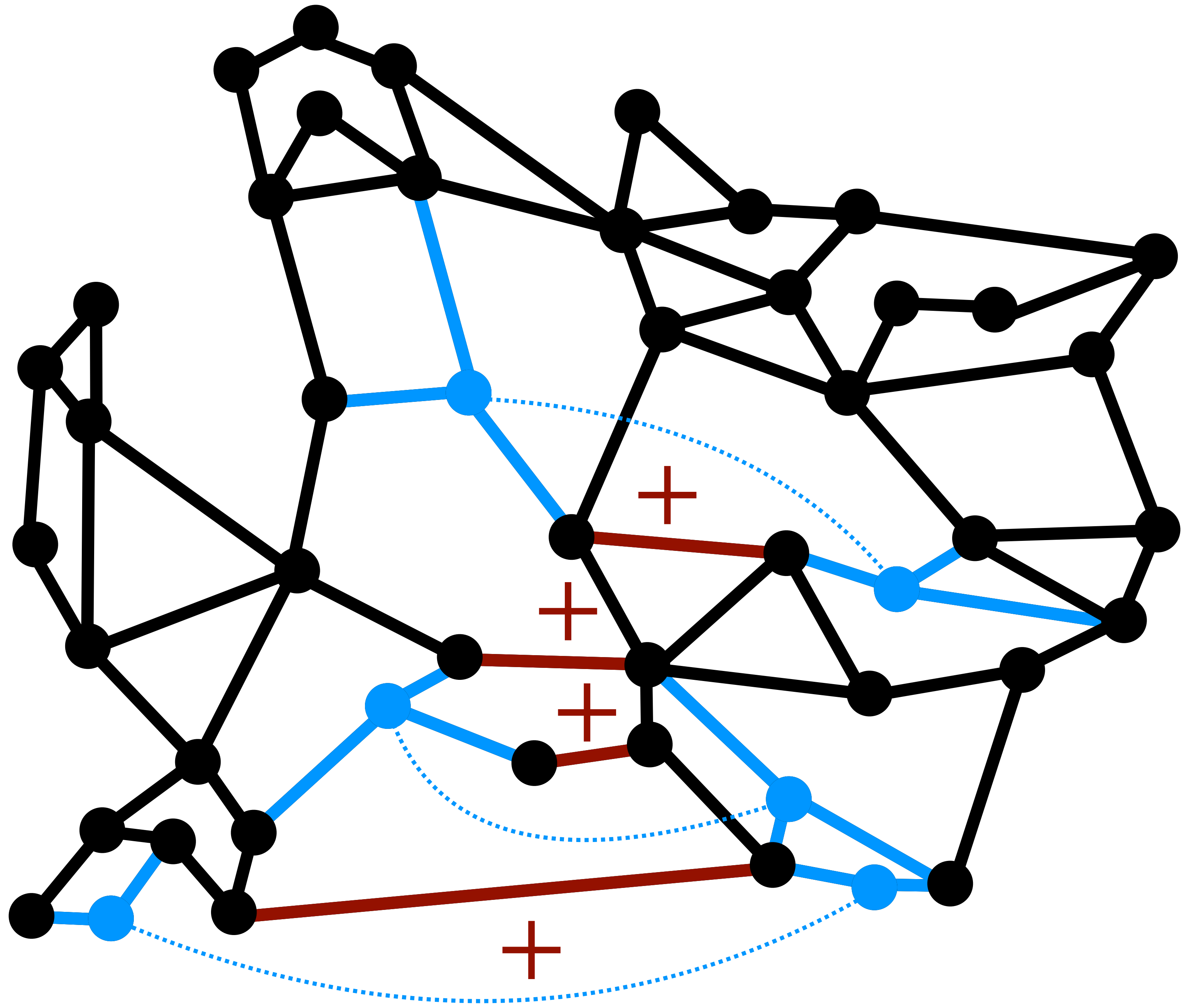


Sparse LC Cut

Sparse Length-Constrained Cuts

Sparse LC Cut Informally.
small total length increase that makes
many close vertex pairs
far

(h, s)-Length ϕ-Sparse Cut Formallyish.
X total length increase for some X that makes
some h-near disjoint vertex pairs w/ degree X/ϕ
at least hs-far

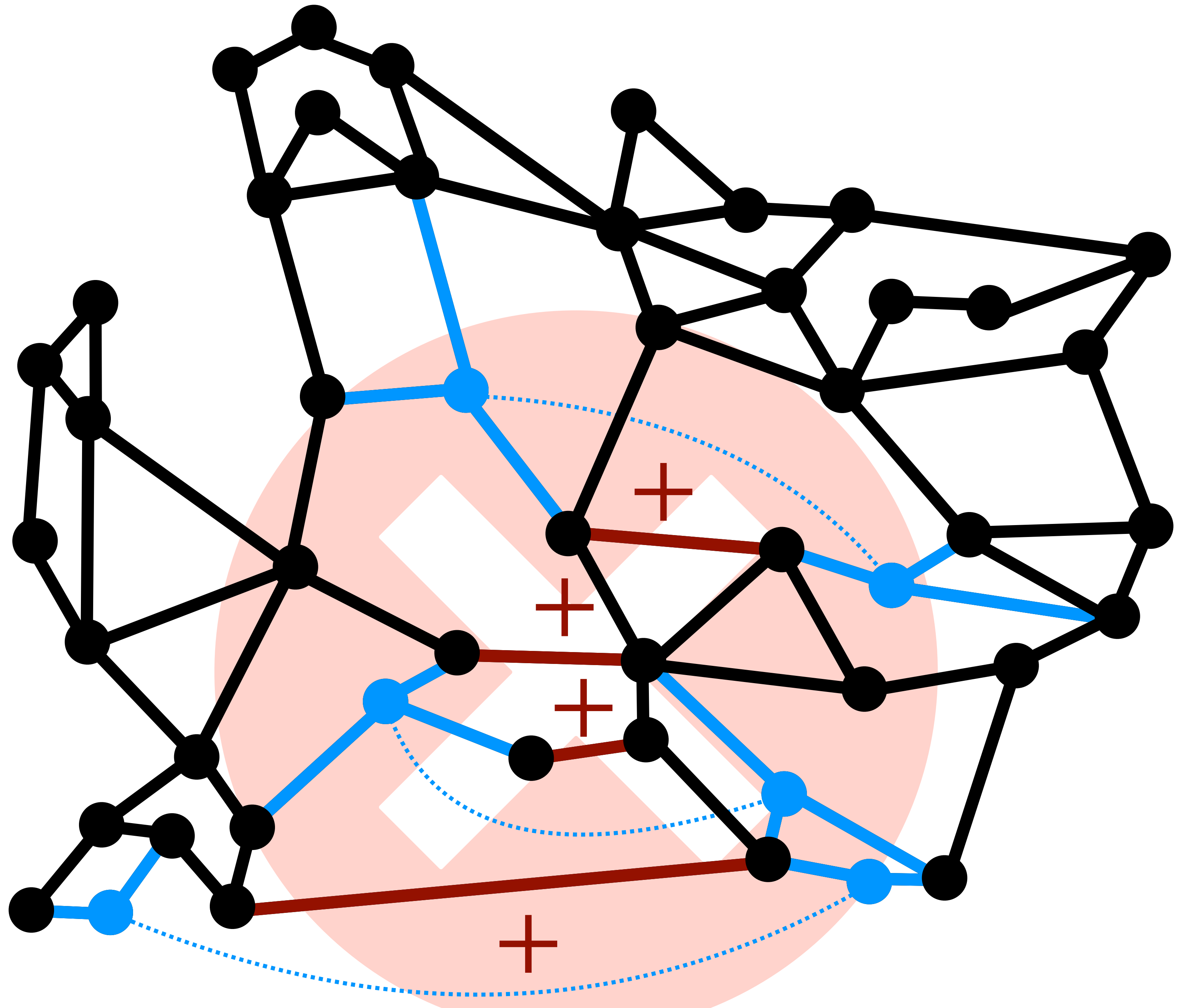


Sparse LC Cut
X length increases
X/ϕ total degree

Note. Any (h, s)-length ϕ-sparse cut has size (i.e. X) at most $\sim\phi m$ (since $\sim X/\phi \leq m$)

Length-Constrained Expanders

Length-Constrained Expanders Informally. hard to make nearby nodes far
(h, s)-**Length ϕ-Expanders Formallyish.** no (h, s)-length ϕ-sparse cuts

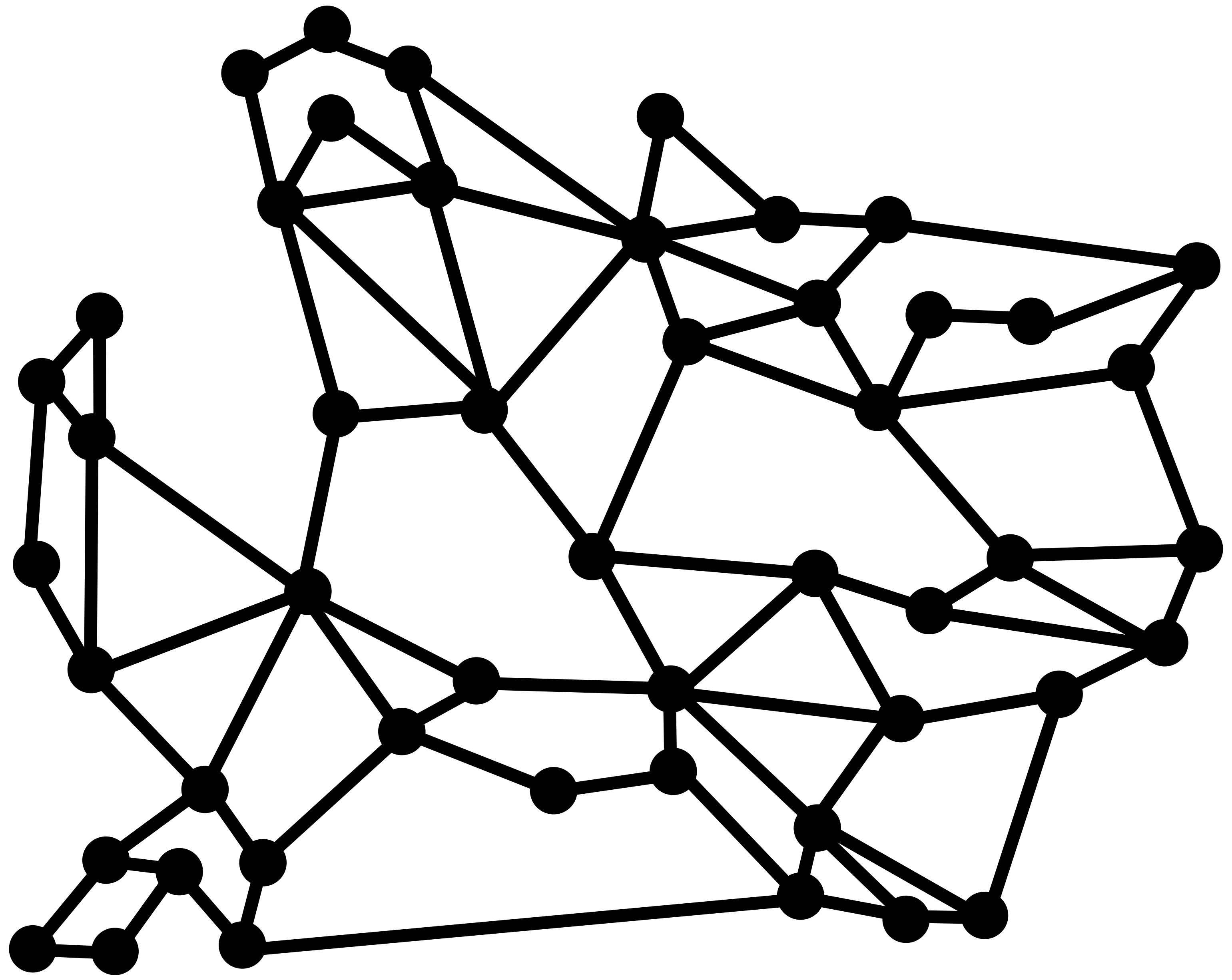


LC Expander

Flow View Informally. Easy for nearby nodes to send flow over short paths

Length-Constrained Expander Decompositions

(h, s)**-Length** ϕ**-Expander Decomposition**
length increases that
make graph
an (h, s)-length ϕ-expander



Arbitrary Graph

Length-Constrained Expander Decompositions

(h, s)-**Length** ϕ-**Expander Decomposition**
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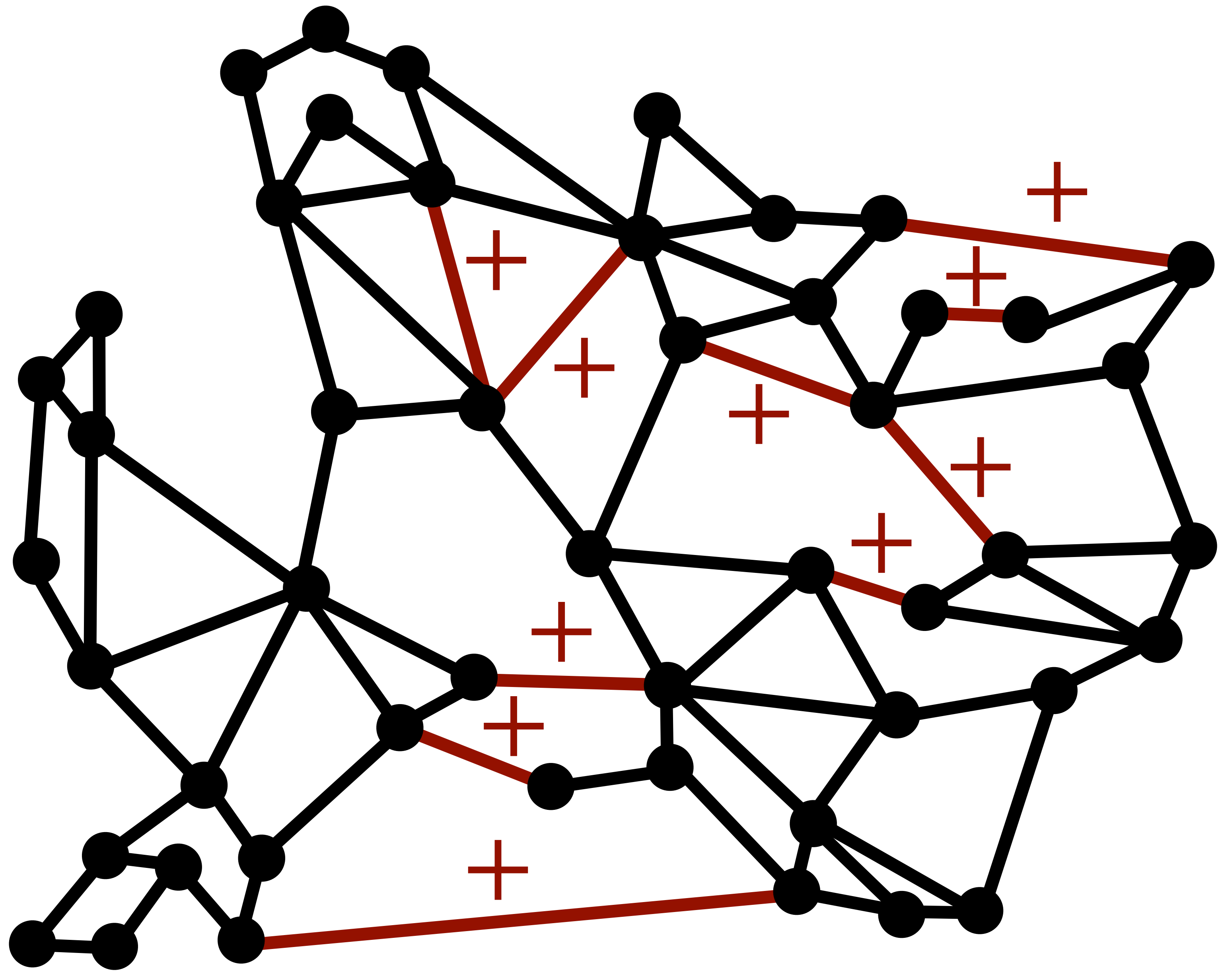

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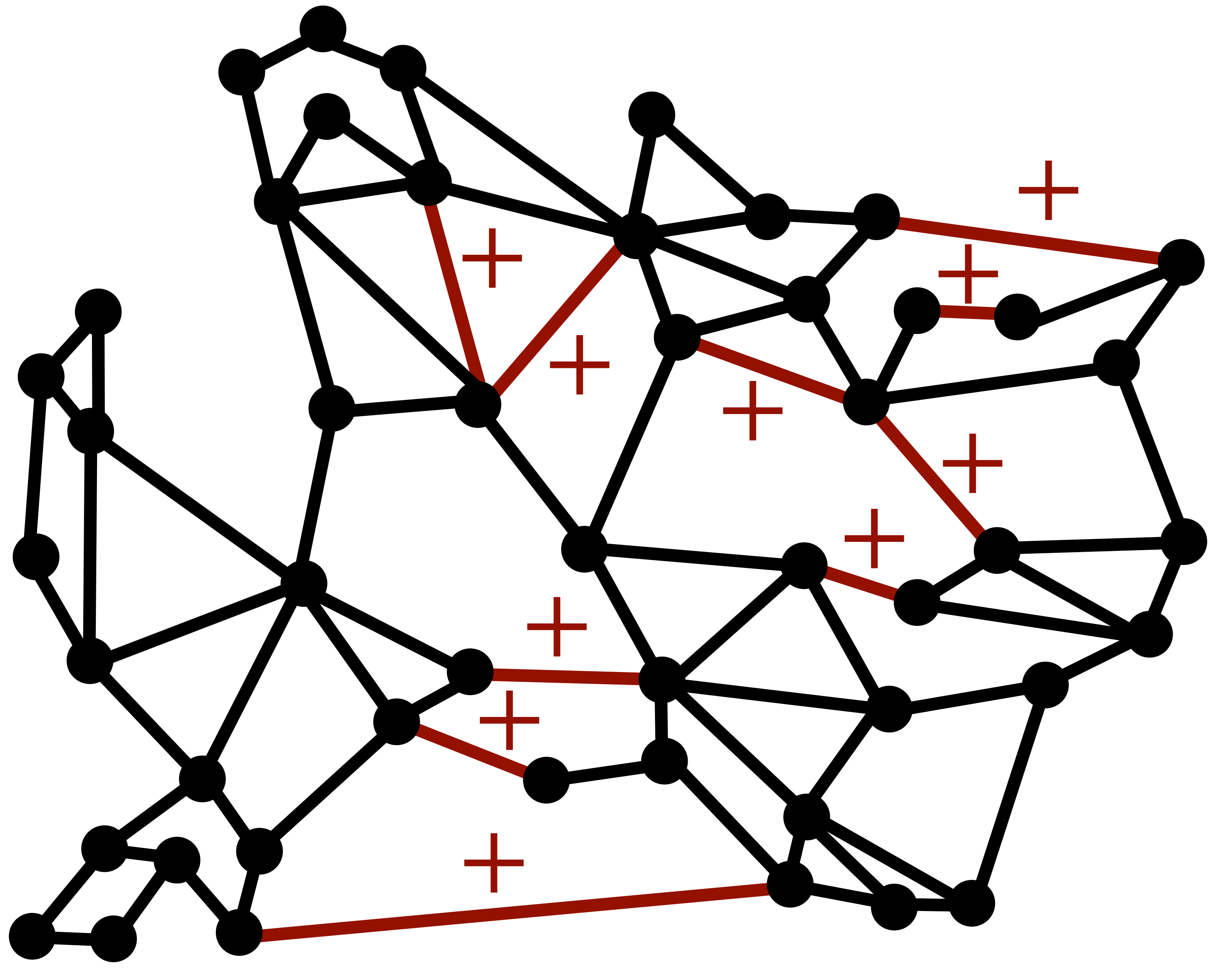


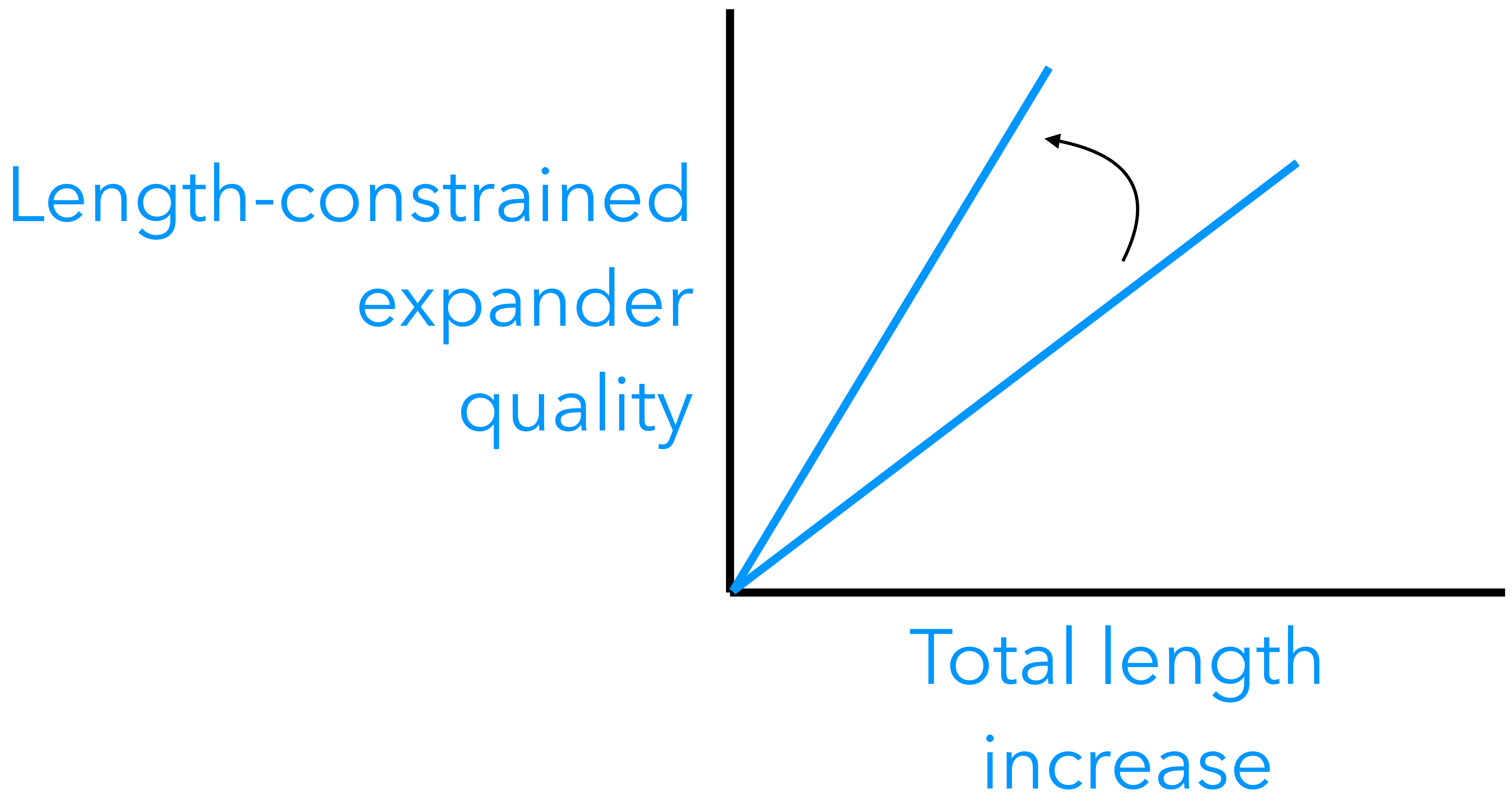
(h, s)-Length ϕ-Expander

Our Result

Our Main Result (Formally)

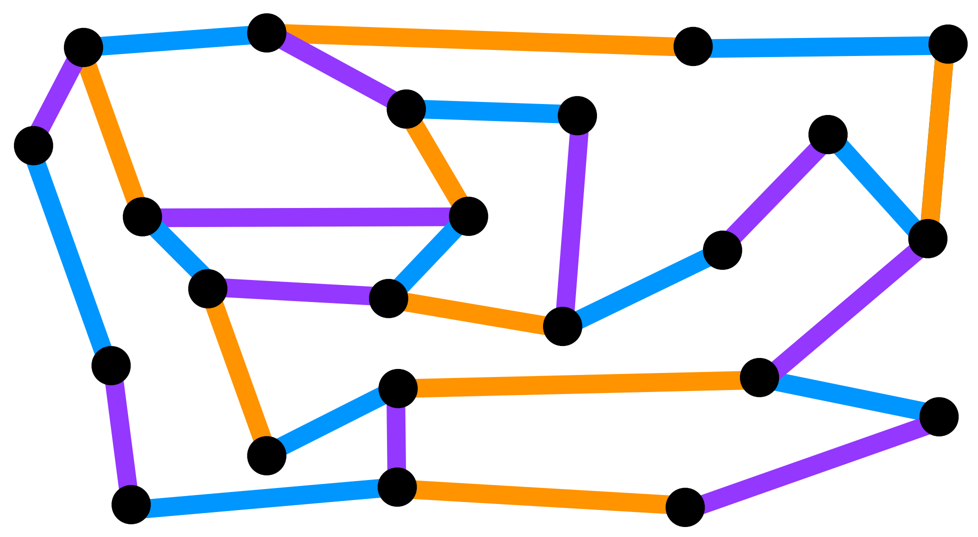
Theorem [BHHT]. Any graph G has an (h, s)-length ϕ-expander decomposition of size $s \cdot n^{O(1/s)} \cdot \phi m$ (proven simply)

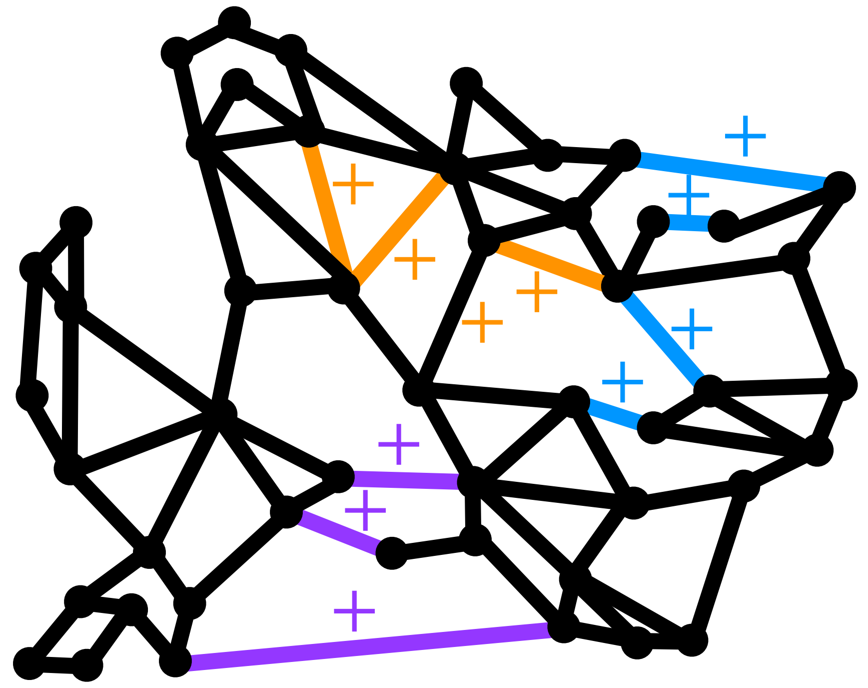
Previously [HHT]. $\log n \cdot s \cdot n^{O(1/s)} \cdot \phi m$ (proven not simply)

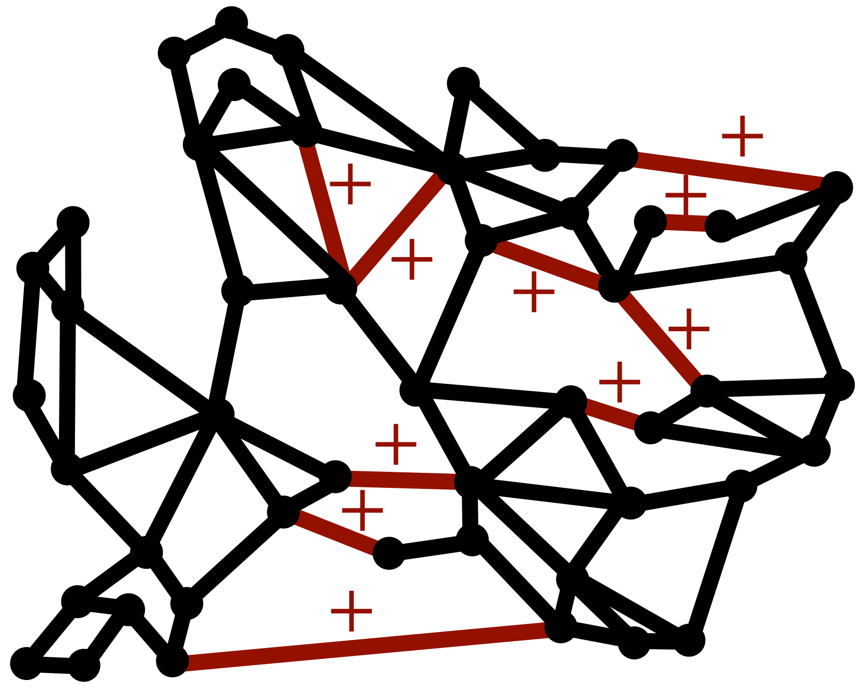


Proof Sketch of Our Result

Outline







Parallel Greedy Arboricity	Easy → (prior)	U of Cuts	Easy →	Existence of LC Decompositions

From ∪ of Cuts to LC Expander Decompositions

Theorem [BHHT]. Any graph G has an (h, s)-length ϕ-expander decomposition of size $s \cdot n^{O(1/s)} \cdot \phi m$ (proven simply)

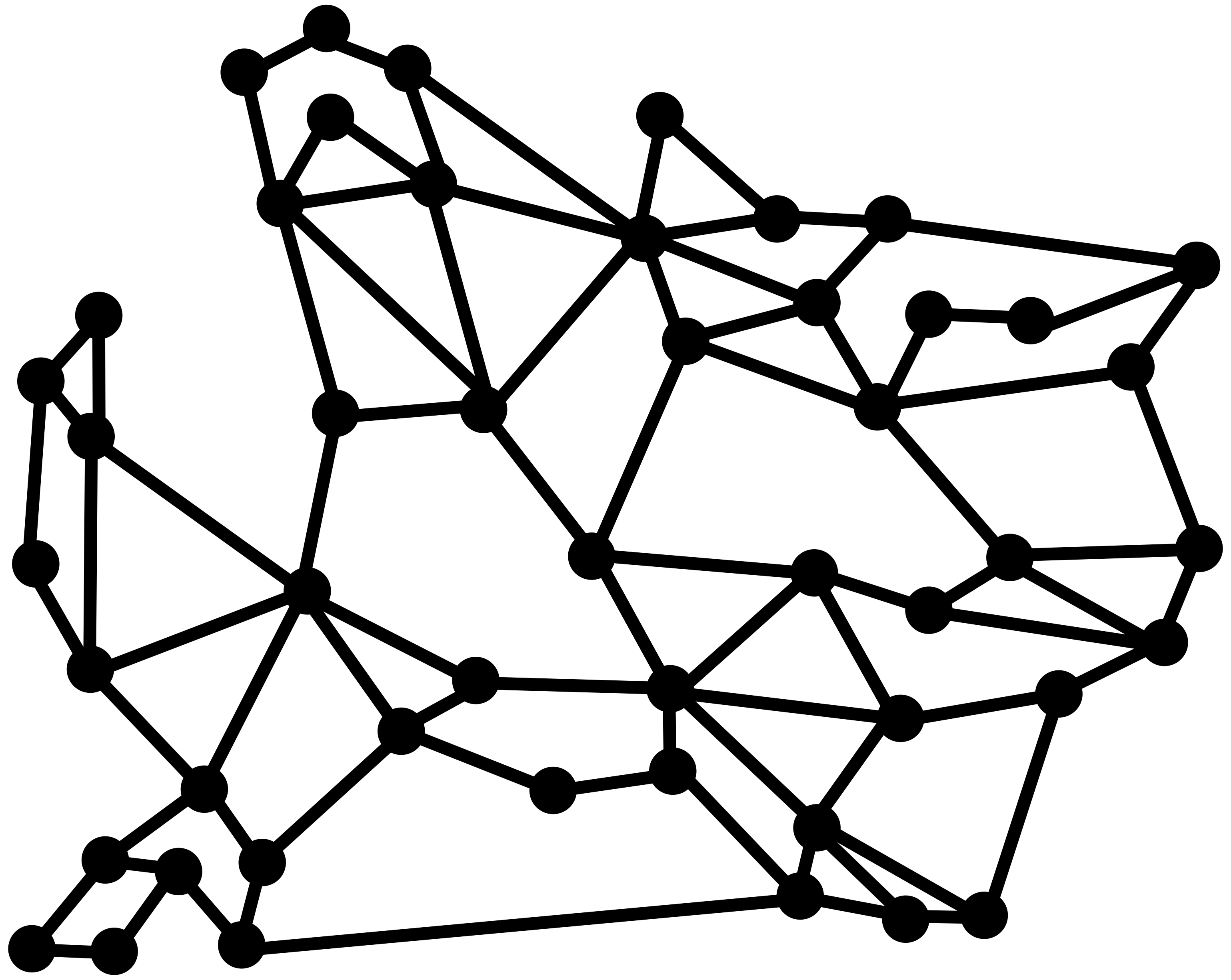
$i \leftarrow 0$, $G_0 \leftarrow G$

While G_i has an (h, s)-length ϕ-sparse cut C_i

$G_{i+1} \leftarrow G_i$ with C_i applied

$i \leftarrow i + 1$

Return $C = \sum_i C_i$



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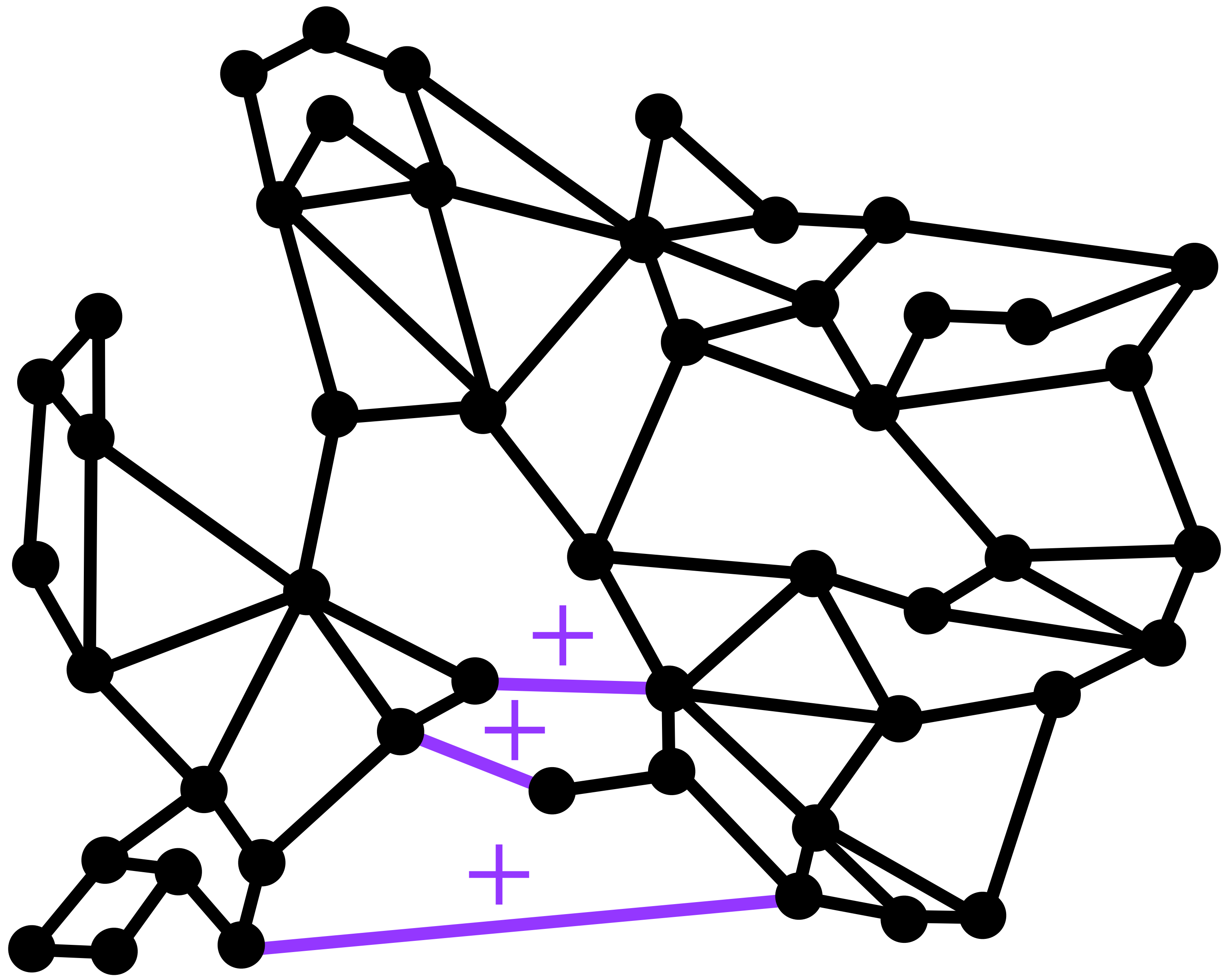

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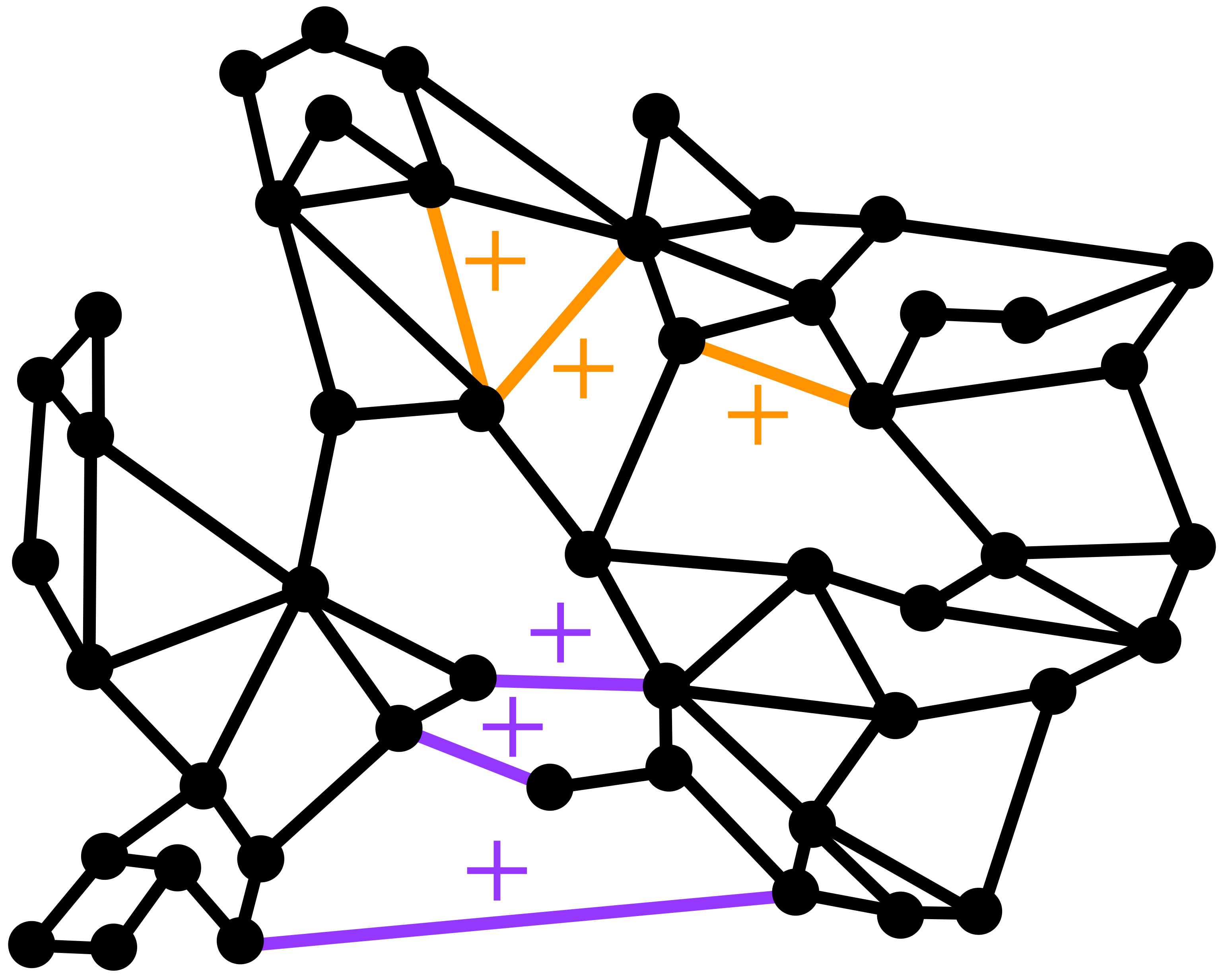

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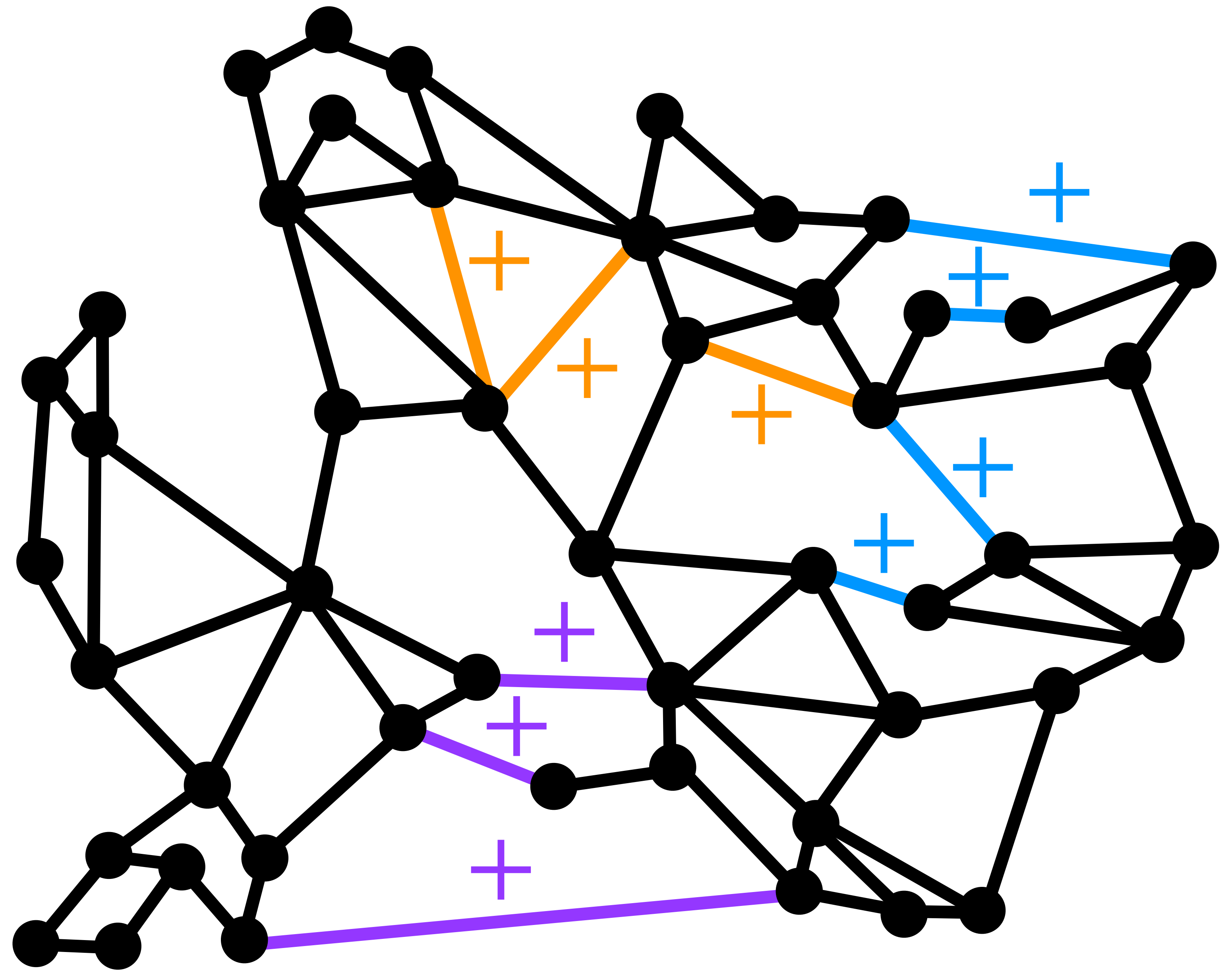

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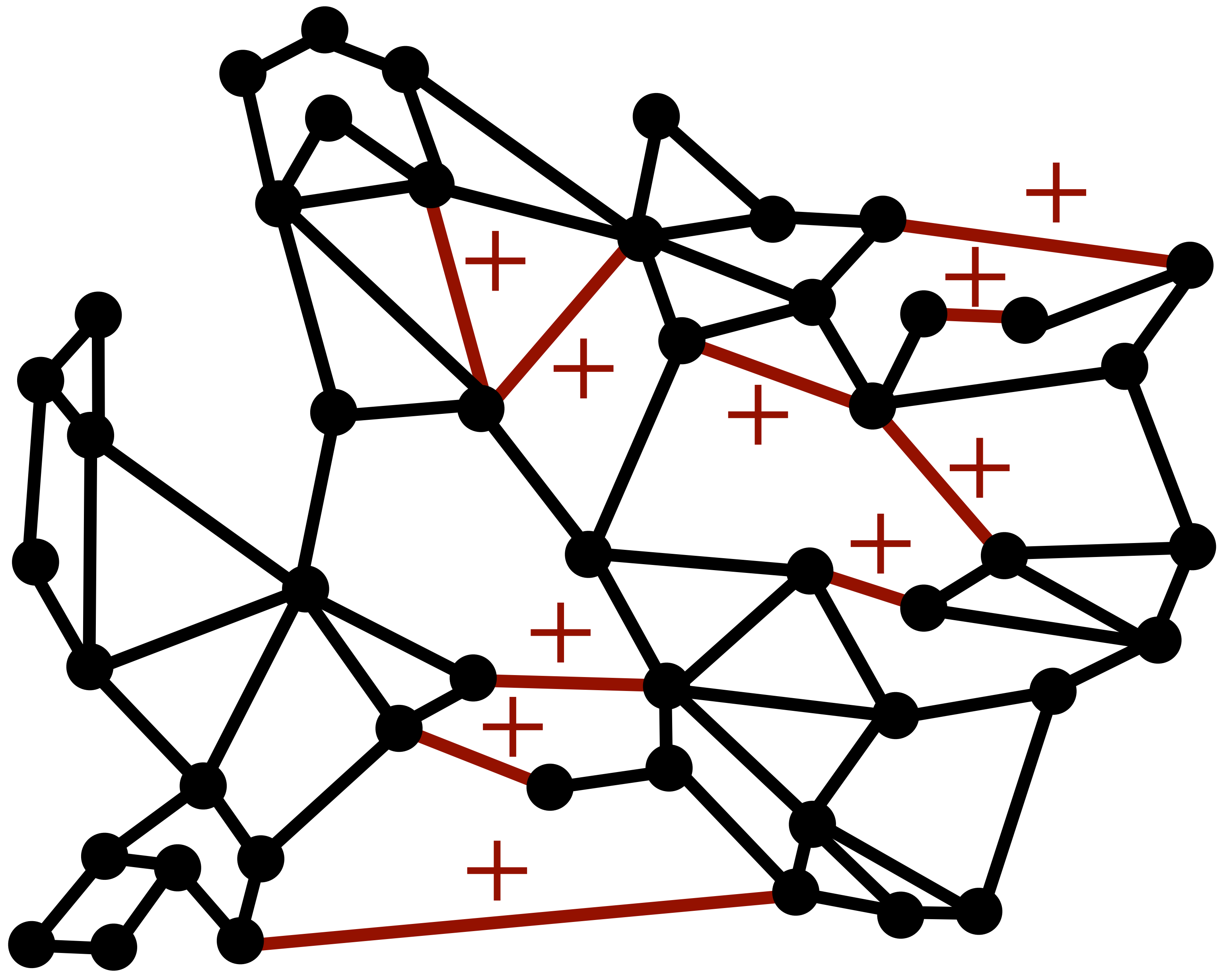

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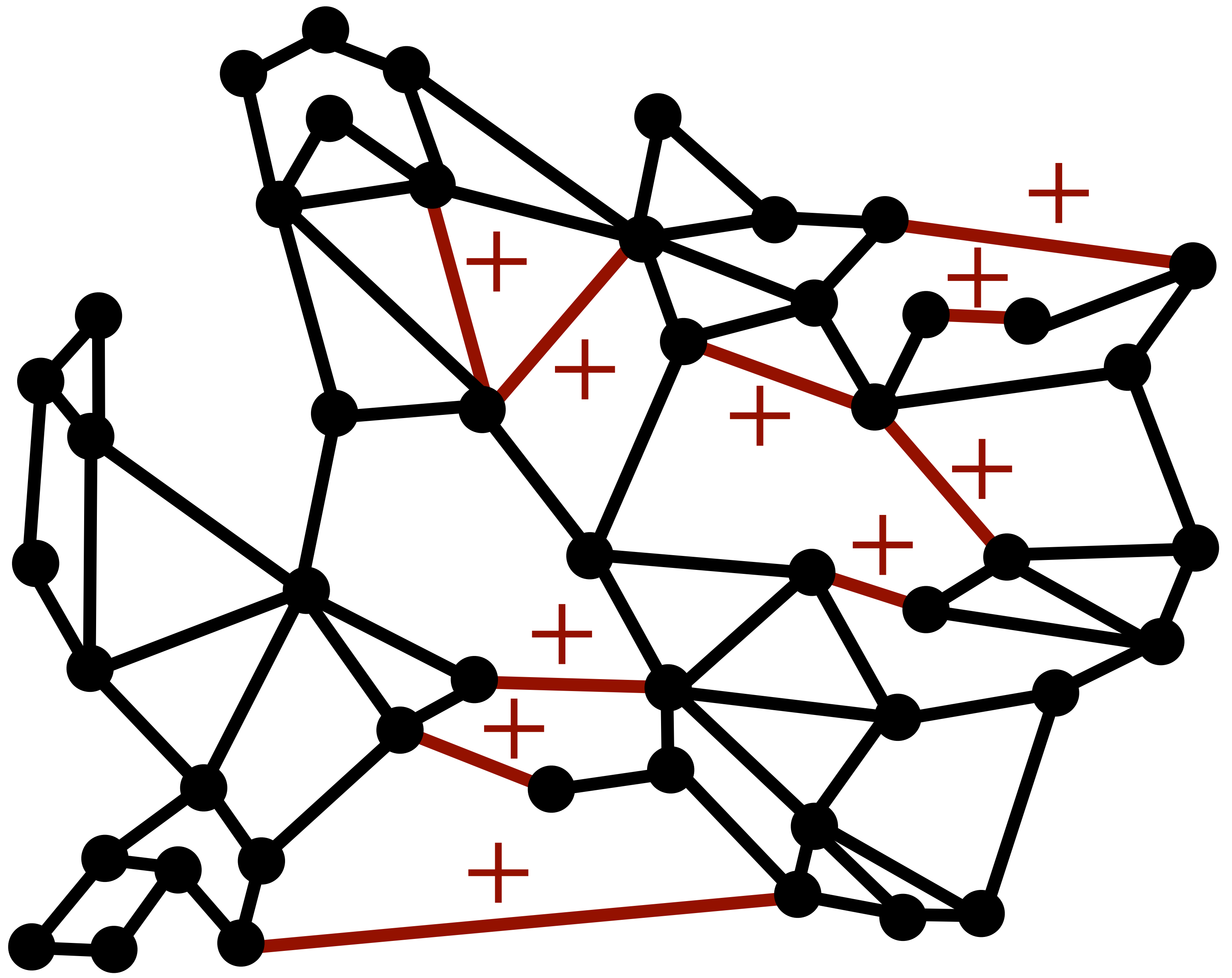
Theorem [BHHT]. Any graph G has an (h, s)-length ϕ-expander decomposition of size $s \cdot n^{O(1/s)} \cdot \phi m$ (proven simply)

$\cdots$ Return $C = \sum_i C_i$

C is an (h, s)-length ϕ-ED (no sparse cuts left)

🤔 But why is it small? 🤔

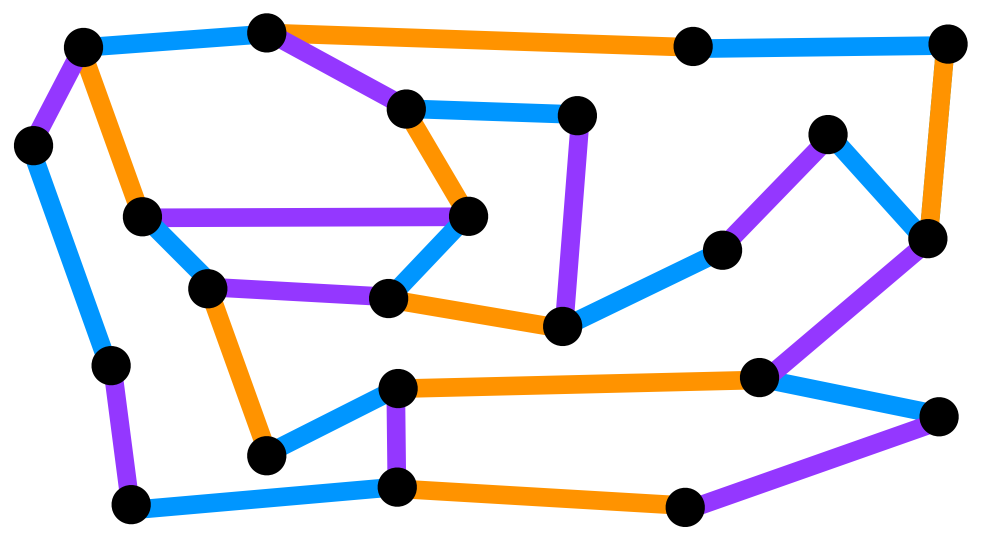


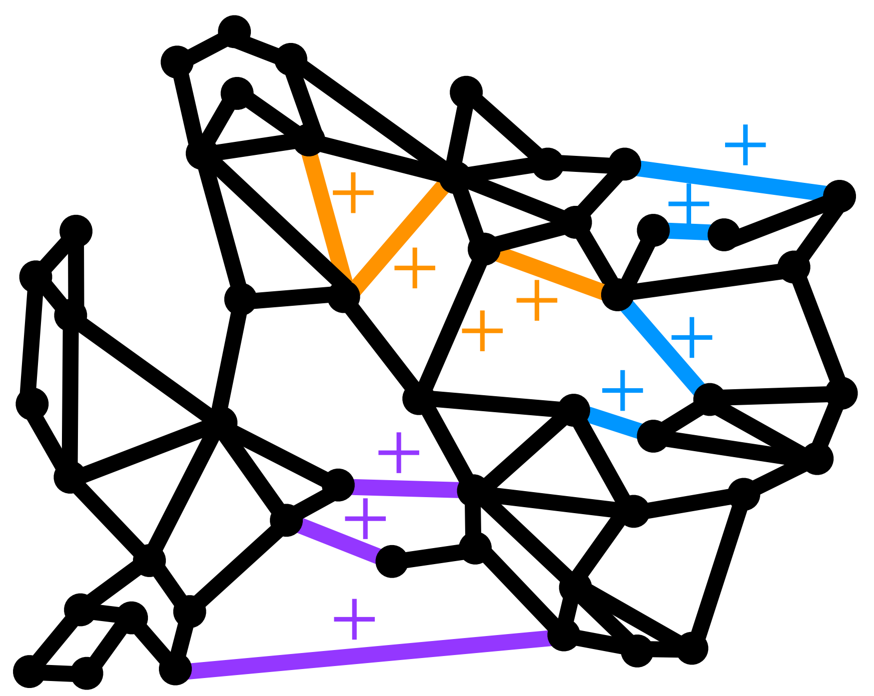
∪ of Cuts [BHHT]. C is an $\sim(h, s)$-length $(sn^{O(1/s)} \cdot \phi)$-sparse cut

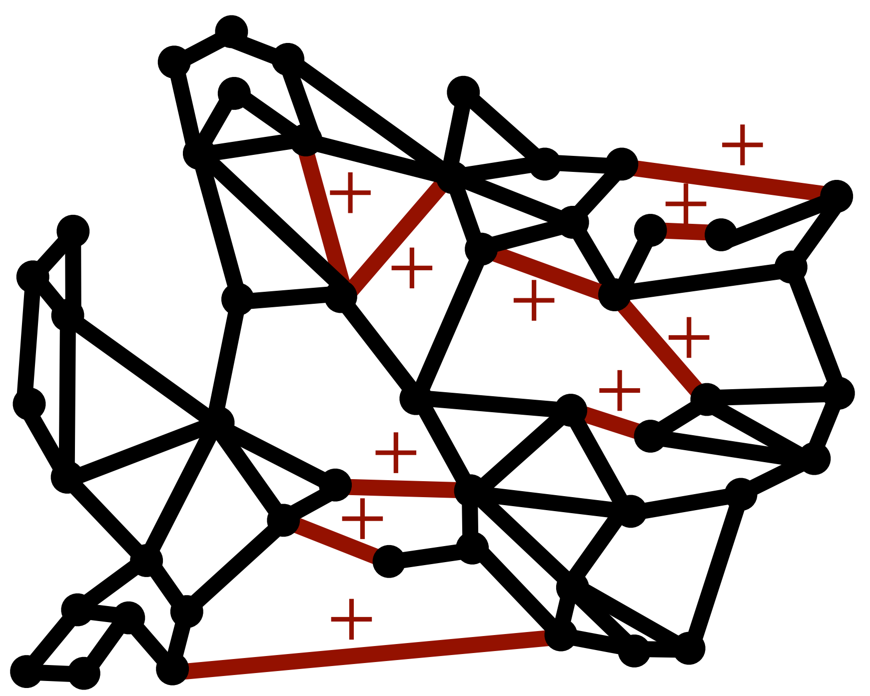
Any (h, s)-length $(sn^{O(1/s)} \cdot \phi)$-sparse cut has size at most $s \cdot n^{O(1/s)} \cdot \phi m$

So C has size at most $s \cdot n^{O(1/s)} \cdot \phi m$

Outline



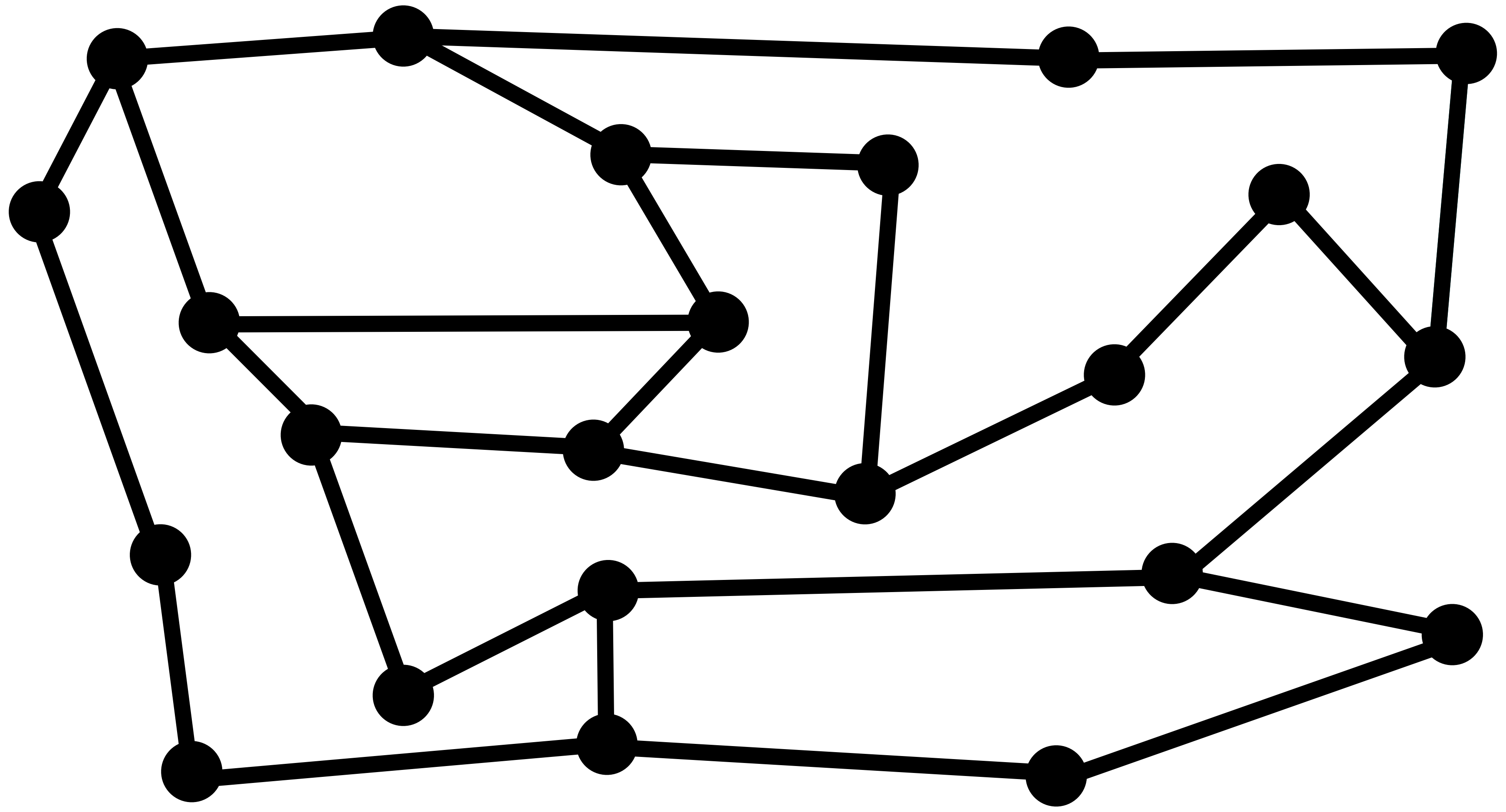




Parallel Greedy Arboricity —Easy (prior)→ U of Cuts —Easy→ Existence of LC Decompositions

From Parallel Greedy Arboricity to ∪ of Cuts

$G = (V, E)$ is an s**-parallel-greedy graph** if its edges decompose into matchings $E = M_1 \sqcup M_2 \sqcup \ldots$ where if $\{u, v\} \in M_i$ then u and v at least s-far in $\left(V, \bigcup_{j<i} M_j\right)$



12-parallel-greedy

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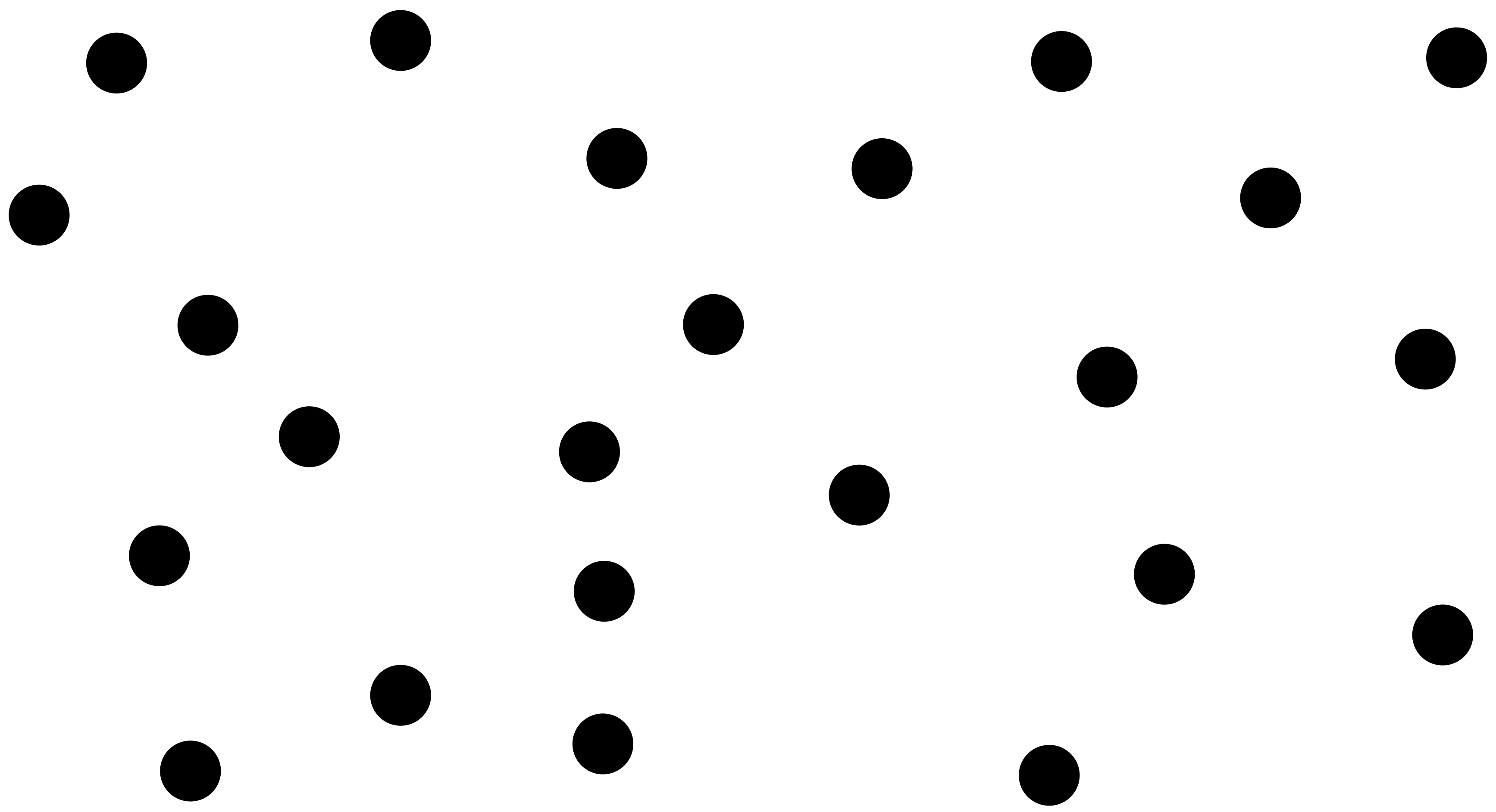

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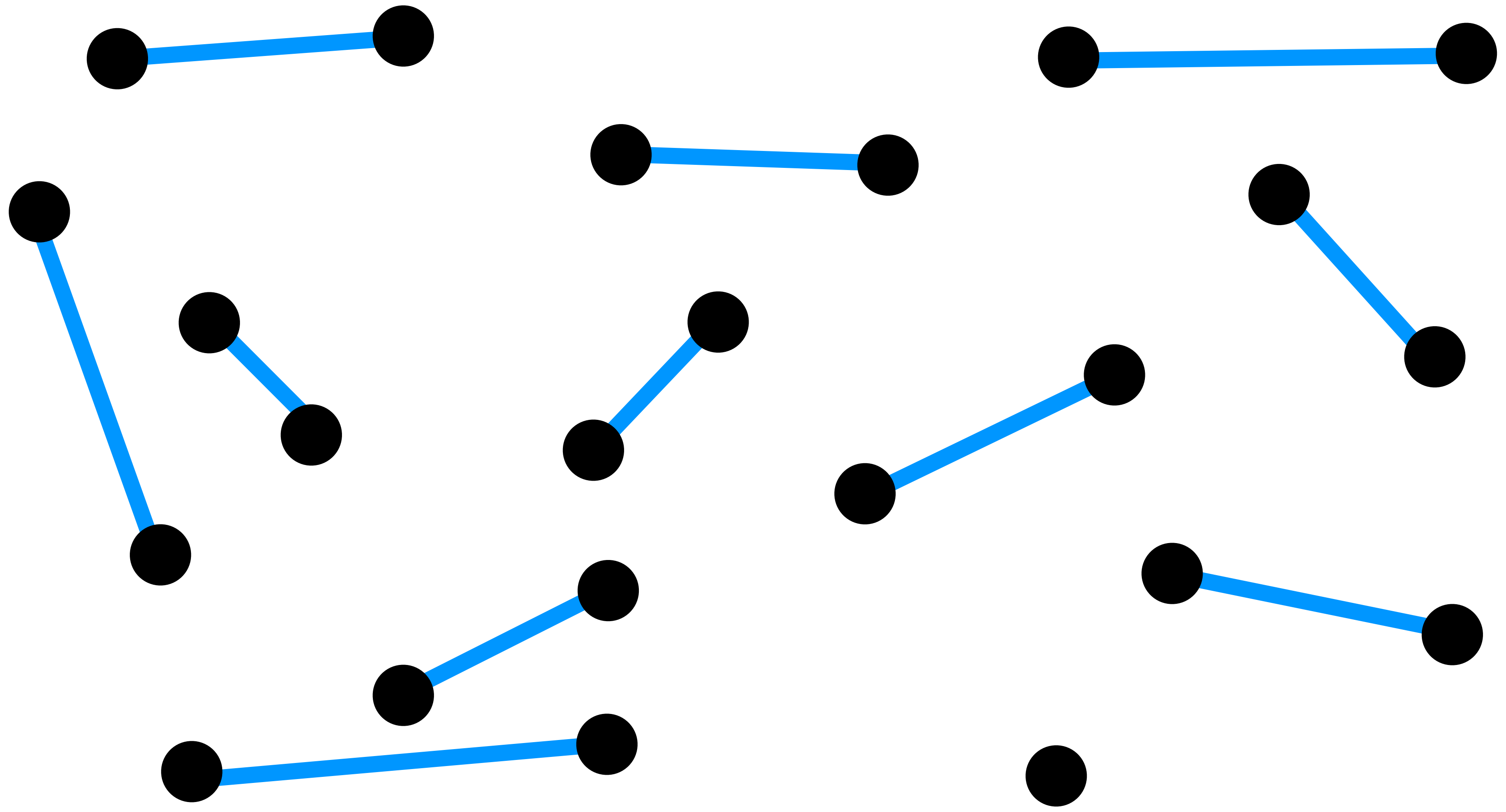

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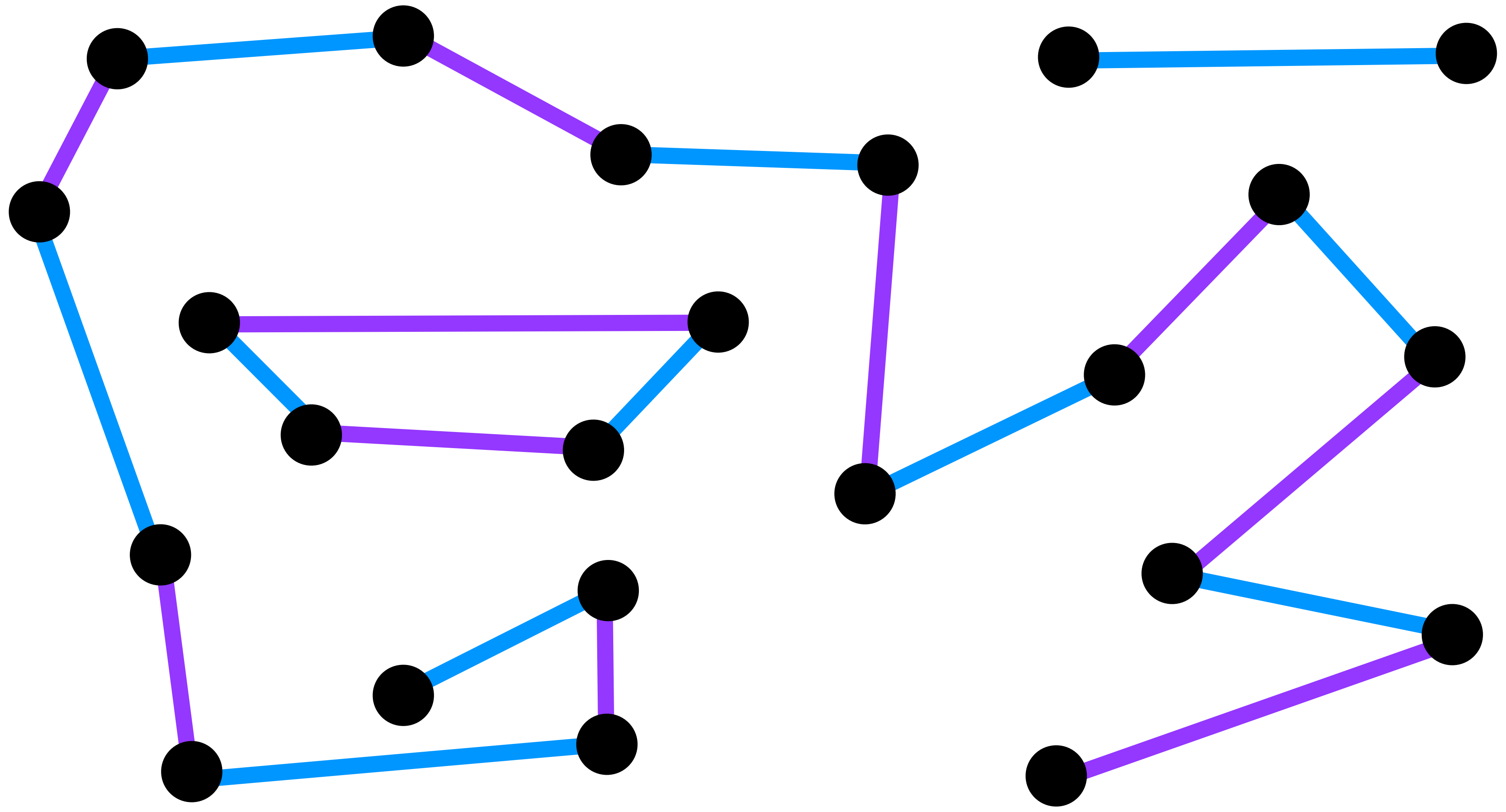

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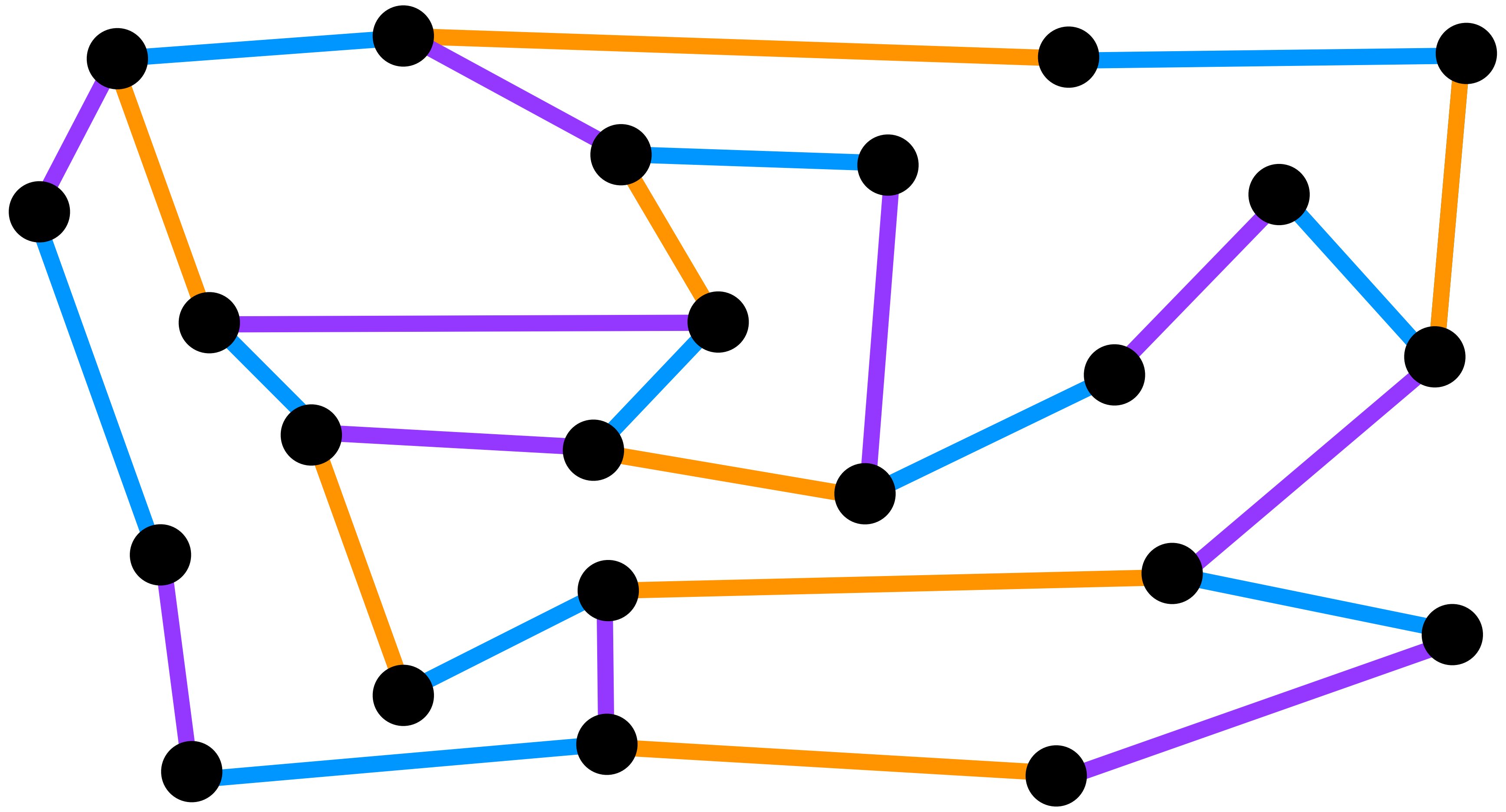
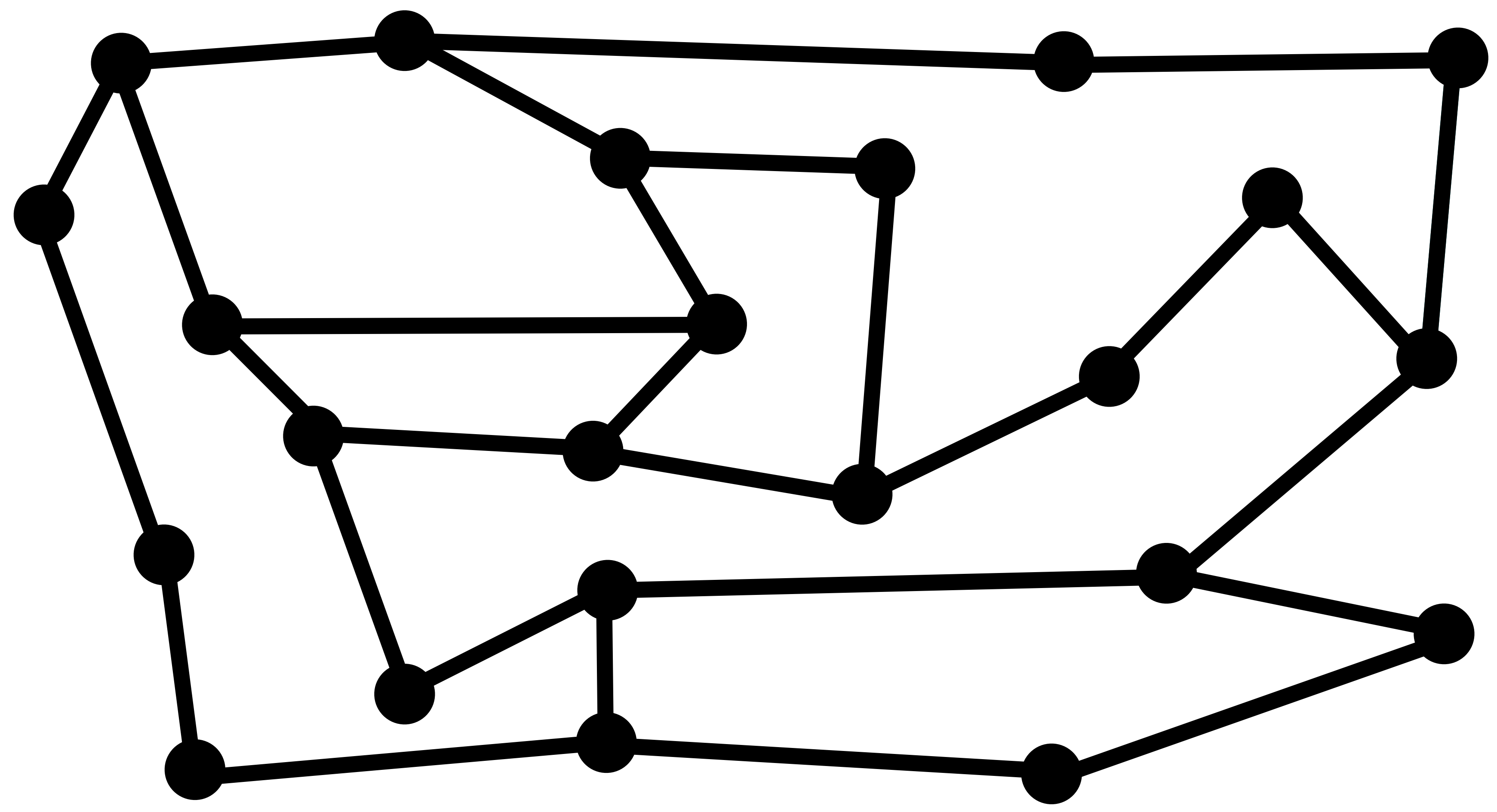


12-parallel-greedy

From Parallel Greedy Arboricity to ∪ of Cuts

The **arboricity** of a graph is the minimum number of forests needed to cover all edges



Arboricity 2

From Parallel Greedy Arboricity to ∪ of Cuts

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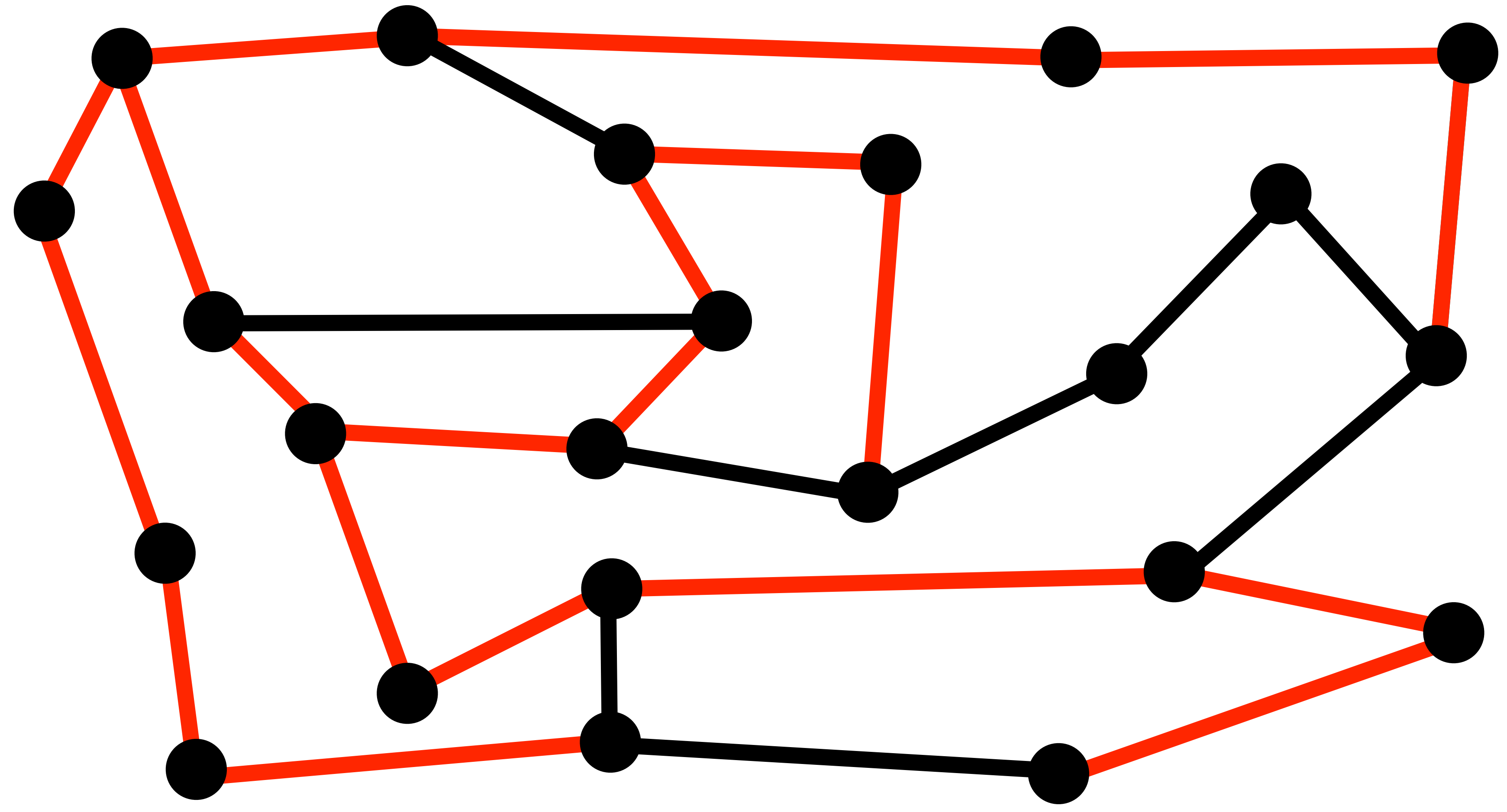

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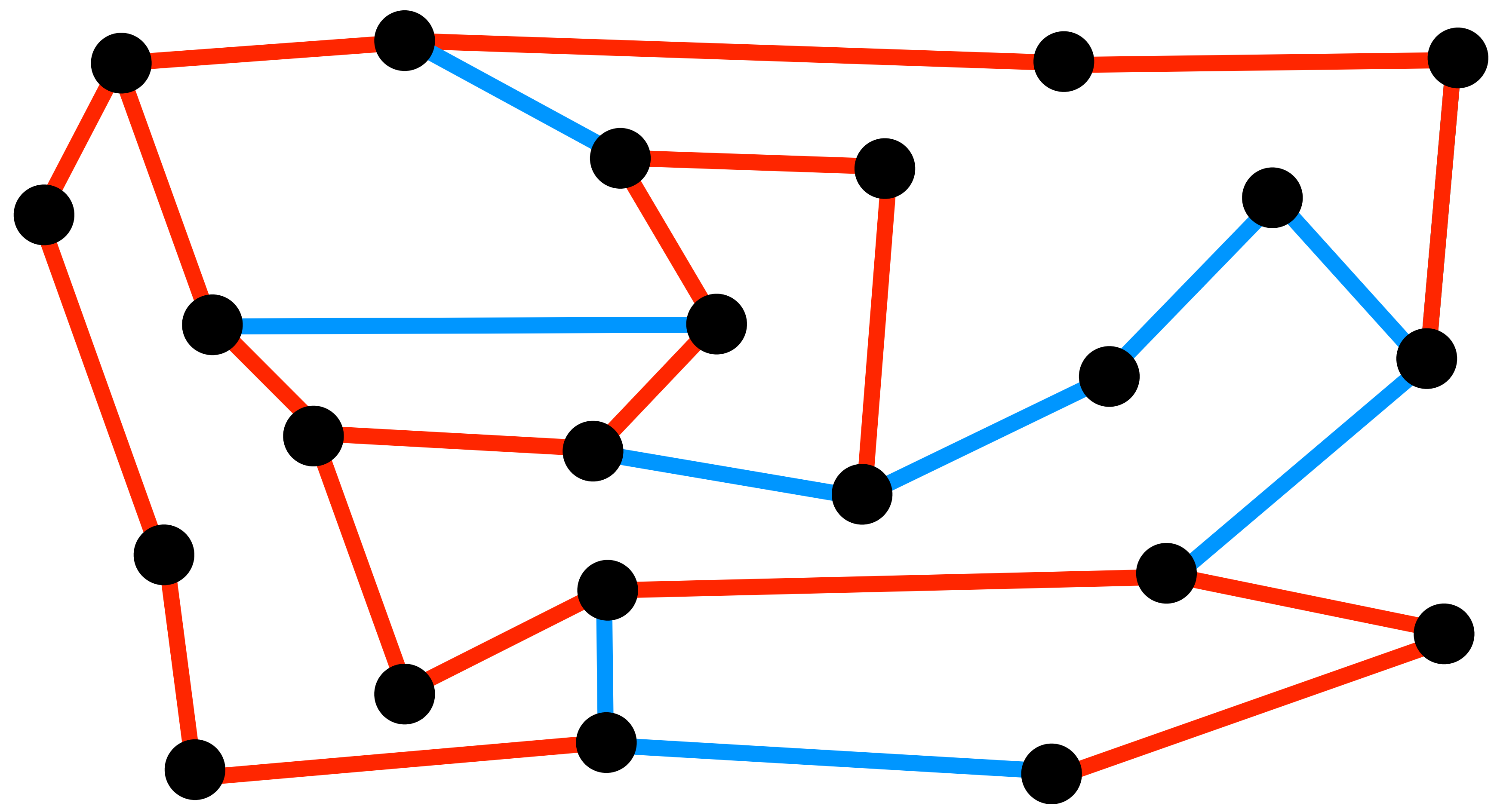


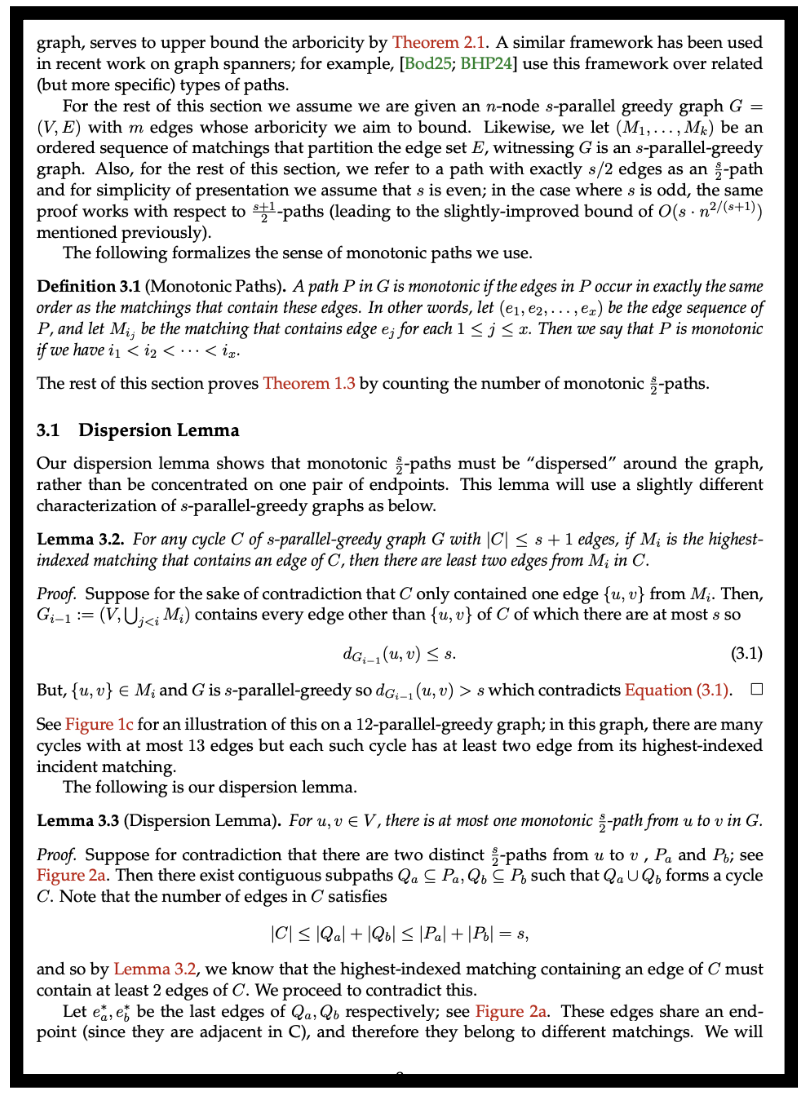
Arboricity 2

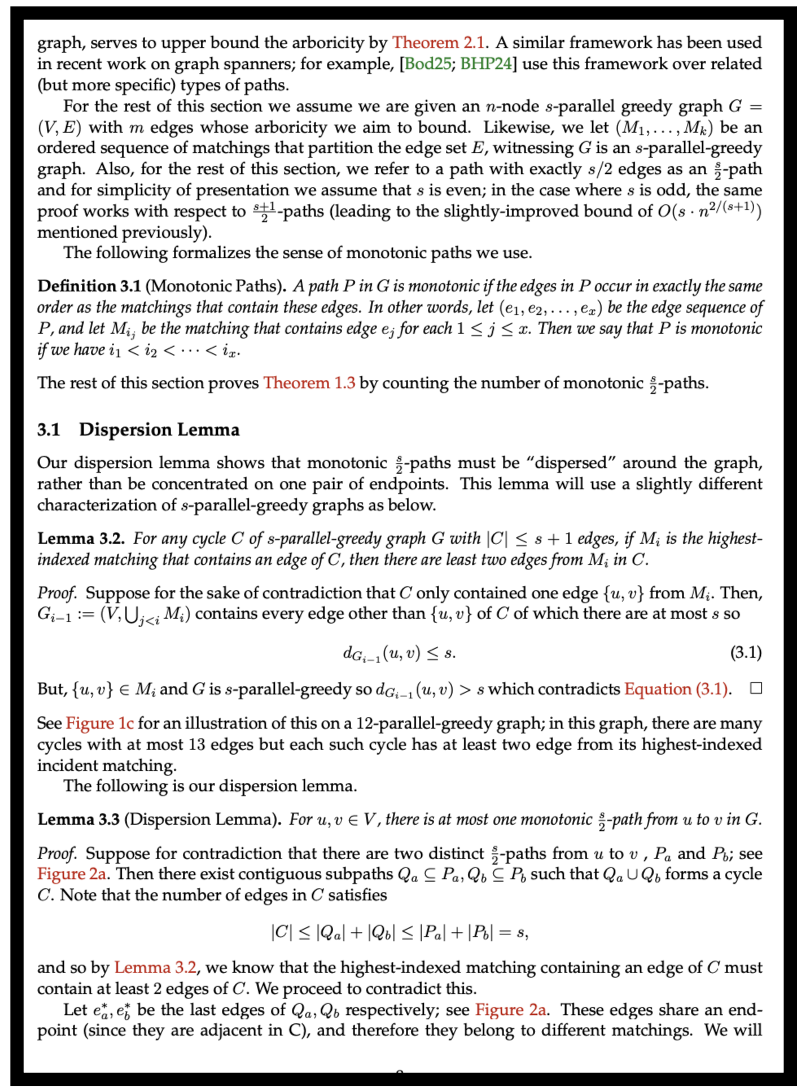
PG Arboricity to ∪ of Cuts [HHT]. C is (h, s)-length $\alpha\phi$-sparse in G where α is the arboricity of s-parallel-greedy graphs

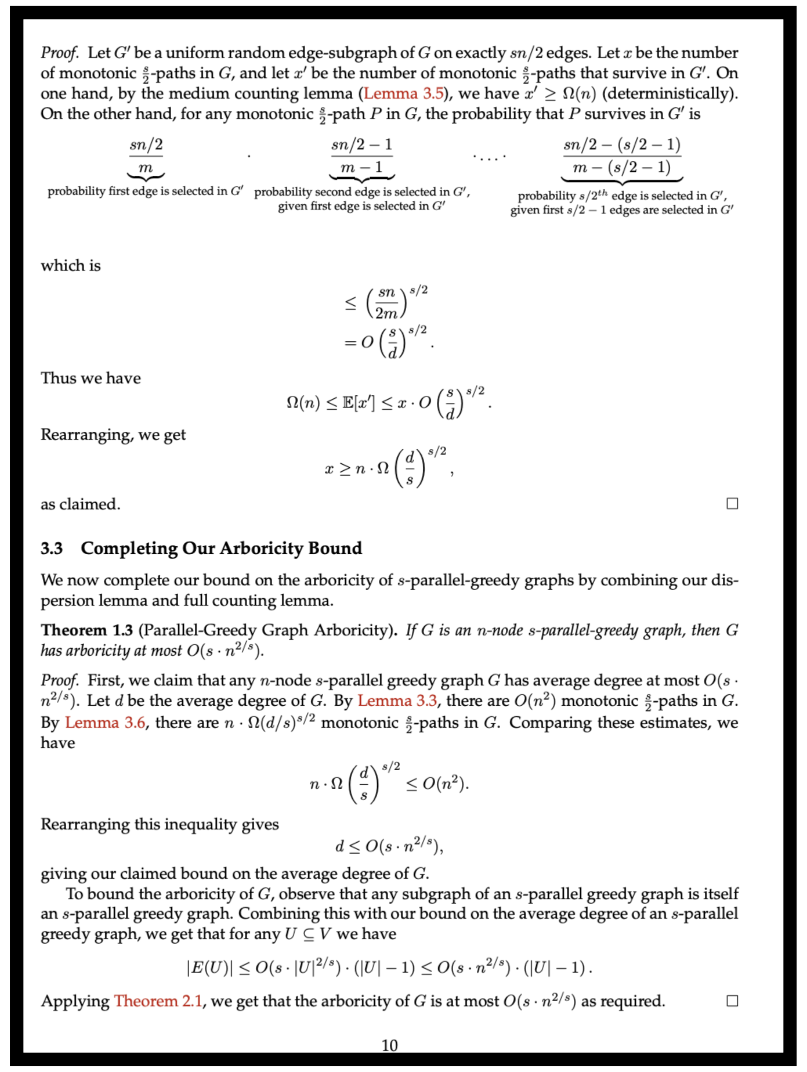
From Parallel Greedy Arboricity to ∪ of Cuts

Parallel Greedy Arboricity [BHHT]. $\alpha \leq s \cdot n^{O(1/s)}$
where α is the arboricity of s-parallel-greedy graphs

~3 pages based on "dispersion/counting" framework.



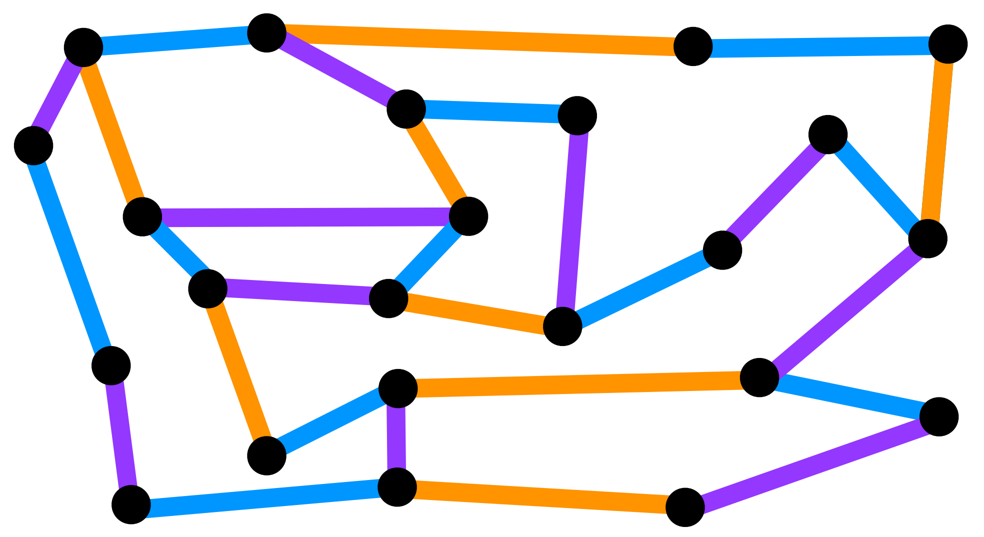


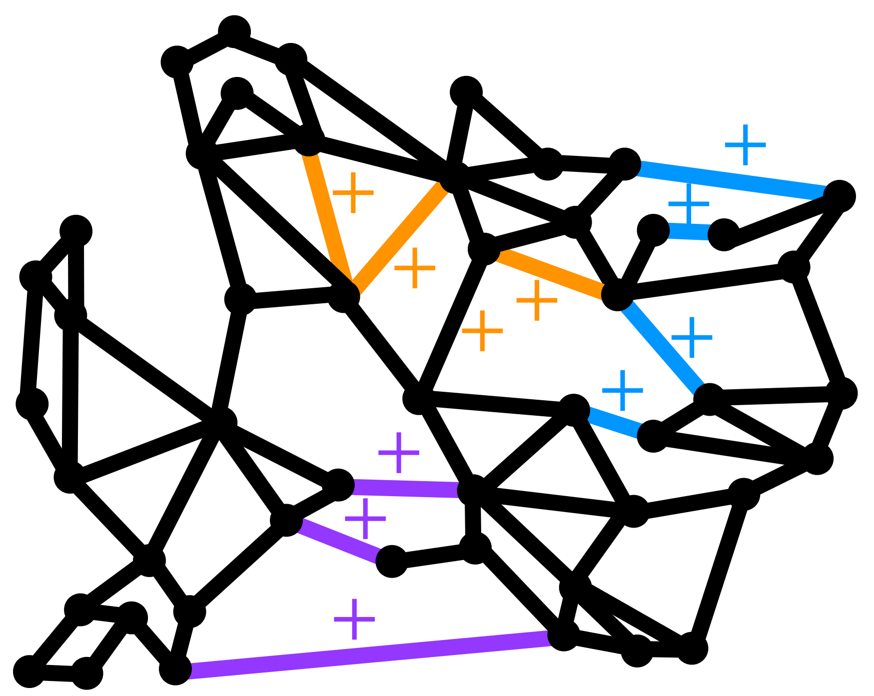


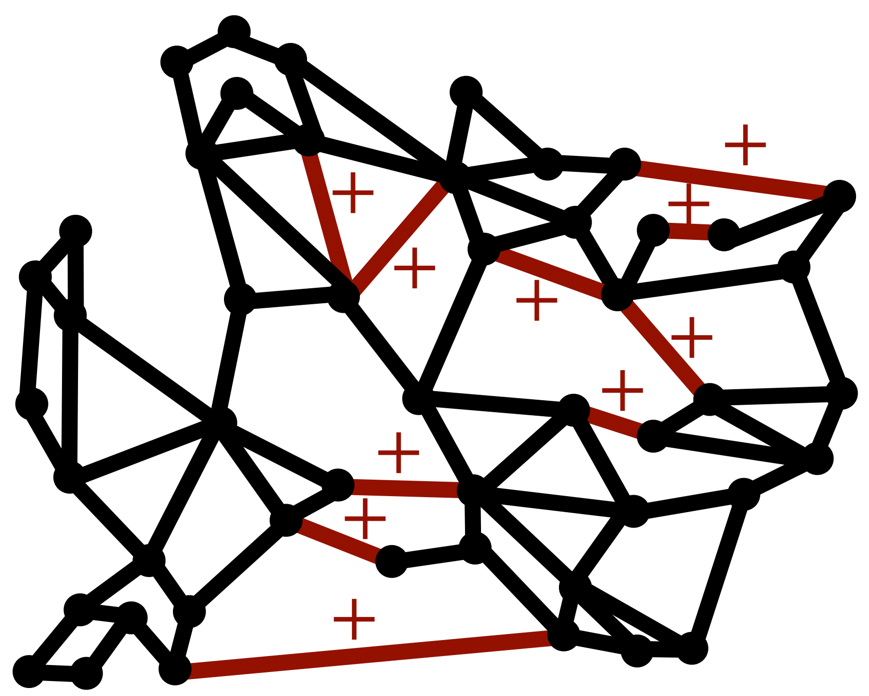
Summarizing

Thanks!

Summary







Parallel Greedy Arboricity → Easy (prior) → ∪ of Cuts → Easy → Existence of LC Decompositions