An Algorithmist's Toolkit — Brown CSCI 2952T – Spring 2026

Lecture 6: LPs—the Geometric View, LP Integrality

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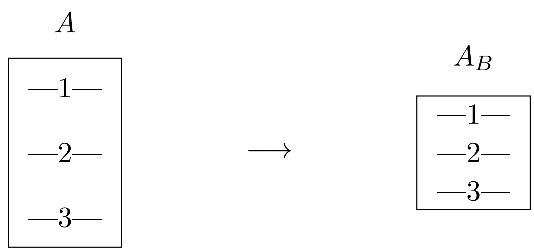
1 Linear Algebra View of LP

1.1 Moving Around in LPs

Definition 1. *An $x \in K$ where $K = \{Ax \leq b\}$ is* ***tight*** *for constraint $\langle a_i, x \rangle \leq b_i$ if and only if $\langle a_i, x \rangle = b_i$. We also define*

$$\mathit{Tight}_A(x) \coloneqq \{a_i \in \text{rows}(A) : \langle a_i, x \rangle = b_i\}$$

For $B \subseteq \text{rows}(A)$, let A_B be the matrix with rows in B.



We will consider A_T, the matrix of tight constraints of A.

Proof to be proved later:

Lemma $(*)$**.** *Given $x \in K, w \in \ker(A_T), \exists \epsilon > 0$ such that*

$$\{x \pm \epsilon \cdot w\} \ \in K$$

Furthermore, if $\langle w, a_i \rangle \neq 0$ for some $a_i \in \text{rows}(A)$, then $\mathit{Tight}_A \subsetneq \mathit{Tight}_A$ for $y \in \{x \pm \epsilon \cdot w\}$.

Lemma 1. *Given $x \in K, w \in \ker(A_T)$ with $\langle w, a_i \rangle \leq 0$. $\forall a - I \in \text{rows}(A)$, we have*

$$x + \epsilon w \in K \text{ and } \mathit{Tight}_A(x) \leq \mathit{Tight}_A(x + \epsilon w) \quad \forall \epsilon \geq 0$$

Proof. We know that $x + \epsilon w \in K$ because $\forall a_i \in \text{rows}(A)$,

$$\begin{aligned} \langle x + \epsilon w, a_i \rangle &= \langle x, a_i \rangle + \underbrace{\epsilon \langle w, a_i \rangle}_{\leq 0} \\ &\leq \langle x, a_i \rangle \\ &\leq b_i \end{aligned}$$

We also know that $\mathrm{Tight}_A(x) \subseteq \mathrm{Tight}_A(x+\epsilon w)$ because $\forall a_i \in \mathrm{Tight}_A(x)$, $\langle a_i, w\rangle = 0$ since $w \in \ker(A_T)$. So

$$\begin{aligned}
\langle a_i, x+\epsilon w\rangle &= \langle a_i, x\rangle + \epsilon\underbrace{\langle a_i, w\rangle}_{=0} \\
&= \langle a_i, x\rangle = b_i
\end{aligned}$$

Thus $\mathrm{Tight}_A(x) \subseteq \mathrm{Tight}_A(x+\epsilon w)$. □

Lemma 2. *Given $x \in K, w \in \ker(A_T)$ with $\langle w, a_i\rangle > 0$ for some $a_i \in \mathrm{rows}(A) \setminus \mathit{Tight}_A(x)$. Then*

$$\exists \epsilon > 0 \ \ s.t. \ \ x + \epsilon' w \in K \quad \forall \epsilon' \in [0, \epsilon]$$

and $\mathit{Tight}_A(x) \subsetneq \mathit{Tight}_A(x+\epsilon w)$.

Proof. Let $I_w =:= \{i : \langle a - i, w\rangle > 0\}$. Let

$$\epsilon_i := \frac{b_i - \langle x, a_i\rangle}{\langle w, a_i\rangle} \text{ for } i \in I_w.$$

Let $\epsilon = \min_{i \in I_w} \epsilon_i$ and $y = x + \epsilon' w$ for $\epsilon' \in [0, \epsilon] > 0$.

Claim: $y \in K$ and $\mathrm{Tight}_A \subset \mathrm{Tight}(y)$.

Consider $a_i \in \mathrm{rows}(A)$. If $a_i \in \mathrm{Tight}_A(x)$, then

$$\langle y, a_i\rangle = \underbrace{\langle x, a_i\rangle}_{=b_i} + \underbrace{\epsilon'\langle a_i, w\rangle}_{=0} = b_i = b_i.$$

Consider $a_i \notin \mathrm{Tight}_A(x)$. If $\langle a_i, w\rangle \le 0$,

$$\langle y, a_i\rangle = \langle x, a_i\rangle + \epsilon'\langle w, a_i\rangle \le \langle x, a_i\rangle \le b_i.$$

If $\langle a_i, w\rangle > 0$. Then

$$\begin{aligned}
\langle y, a_i\rangle &= \langle x, a_i\rangle + \epsilon'\langle w, a_i\rangle \\
&\le \langle x, a_i\rangle + \epsilon\langle w, a_i\rangle && \epsilon' \in [0, \epsilon] \\
&\le \langle x, a_i\rangle + \epsilon_i\langle w, a_i\rangle && \epsilon = \min \epsilon_i \\
&= b_i
\end{aligned}$$

If we choose $\epsilon' = \epsilon$ and $\exists i$ such that $\epsilon = \epsilon_i$ (attains minimum value), then both become equal and we end up with a new tight constraint. Thus $\mathrm{Tight}_A(x) \subsetneq \mathrm{Tight}_A(x+\epsilon w)$. □

Proof of Lemma (∗). We say that $w \in \ker(A)$ is type 1 if $\langle w, a_i\rangle \le 0$ for all $a_i \in \mathrm{rows}(A)$. Similarly, we say that $w \in \ker(A)$ is type 2 if $\langle w, a_i\rangle > 0$ for some $a_i \in \mathrm{rows}(A)$ and $a_i \notin \mathrm{Tight}_A(x)$.

Remark 1. $w \in \ker(A) \implies -w \in \ker(A)$*. So $w, -w$ are either type 1 or type 2.*

Suppose they are both type 1 such that $\langle w, a_i\rangle = \langle -w, a_i\rangle = 0 \quad \forall a_i$. Then by Lemma 1, we know that

$$x + \epsilon w \in K \quad \forall \epsilon \ge 0.$$

Now suppose $\langle w, a_i \rangle \neq 0$ for some $a_i \in \text{rows}(A)$. At least one of $w, -2$ is type 2, so without loss of generality suppose w is type 2.

Case 1: $-w$ is type 1. By Lemma 2, we have

$$\epsilon \text{ s.t. } x + \epsilon w \in K \text{ and } \text{Tight}_A(x + \epsilon w) \supsetneq \text{Tight}_A(x).$$

By Lemma 1,

$$x - \epsilon w \in K \quad \forall \epsilon > 0.$$

Case 2: Both $w, -w$ are type 2. By Lemma 2, there exists ϵ_1, ϵ_2 for $w, -w$ respectively such that $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ satisfies

$$x + \pm\epsilon' w \in K \quad \forall \epsilon' \in [0, \epsilon] \qquad \text{and } \text{Tight}_A(x \pm \epsilon w) \supsetneq \text{Tight}_A(x).$$ ☐

1.2 Feasible Solutions

Definition 2. *x is a* ***basic feasible solution*** *of K if and only if $x \in K$ and* $\text{rank}(\textit{Tight}_A(x)) = n$

Claim 1. *If $K = \{x : Ax \leq b, x \geq 0\} \neq 0$, then K has a BFS.*

Proof. Let

$$A' = \begin{pmatrix} A \\ -I \end{pmatrix}, \quad b' = \begin{pmatrix} b \\ 0 \end{pmatrix} \implies K = \{A'x \leq b'\}$$

Let x be any solution maximizing $|\text{Tight}_{A'}(x)| := |T'|$. Assume for the sake of contradiction that x is not a BFS, then $\text{rank}(T') < n$ and $\text{rank}(A'_{T'} < n$. Then by rank nullity,

$$\exists w \neq 0 \text{ s.t. } w \in \ker(A'_{T'})$$

and we also know that $\exists a_i \in \text{rows}(A')$ such that $\langle a_i, w \rangle \neq 0$, since if $w_j \neq 0$, then $\langle -e_j, w_j \rangle \neq 0$. So by lemma $(*)$, $\exists \epsilon > 0$ such that $x \pm \epsilon w \in K$ with $\text{Tight}_A(x) \subsetneq \text{Tight}_A(y)$ for some $y \in \{x \pm \epsilon w\}$, which is a contradiction to the choice of x (maximizing $|T'|$) and thus x must be a BFS. ☐

Lemma 3. *x is a BFS if and only if $x \in K$ and there exists a basis $B \subseteq \text{rows}(A)$ of $\mathbb{R}^n$ with $A_B x = b_B$.*

Proof. ($\Rightarrow$) $x \in K$ by definition. We know that $\text{rank}(\text{Tight}_A(x)) = n$, so $\exists$ a basis B such that

$$B \subseteq \text{Tight}_A(x) \subseteq \text{rows}(A).$$

and B is a basis of $\mathbb{R}^n$. We also know that $A_B x = b_B$ since $B \subseteq \text{Tight}_A(x)$.

($\Leftarrow$) $x \in K$, and since $A_B x = b_B$, we have that the basis $B \subseteq \text{Tight}_A(x)$. So $\text{rank}(\text{Tight}_A(x)) = n$. ☐

Fact: If $B \subseteq \text{rows}(A)$ is a basis of $\mathbb{R}^n$ then $A_b x = b_B$ has either 0 or 1 solution.

This comes from the fact that $\text{rank}(A_B) = n$, so from problem 7 on the linear algebra review we showed that this only has 0 or 1 solution.

Theorem 1. *We can output all BFs in* $m^n \cdot \text{poly}(N)$ *time.*

Algorithm 1 Output All BFSs

$S \geq \emptyset$
for $B \in \text{rows}(A)$ with $|B| = n$ **do**
  if B is a basis and $\exists x$ s.t. $A_B x = b_B$ and $x \in K$ **then**
    $S \leftarrow S \cup \{x\}$
  end if
end for
return S

Proof.
Correctness: If x is a BFS, by lemma 3 $\exists$ basis $B \subseteq \text{rows}(A)$ such that $A_B x = b_B$ and $x \in K$ by the fact means that $x \in S$.

If $x \in S$, then $\exists$ a basis $B \subseteq \text{rows}(A)$ such that $A_B x = b_B$ and $x \in K$, thus x is a BFS.

Runtime: We check if B is a basis in $\text{poly}(N)$ time from problem 8 on the linear algebra homework.

We find x such that $A_B x = b_B$ in $\text{poly}(N)$ time from fact 3.3 on the linear algebra review.

We have a total of

$$\binom{m}{n} \leq \left(\frac{me}{n}\right)^n \leq m^n$$

iterations. □

Theorem 2. *Can solve LP feasibility in* $\text{poly}(N) \cdot m^{2n}$ *time.*

Proof sketch. We first put the LP in the form of

$$L = \{x : Ax \leq b, x \geq 0\}$$

which doubles n. Let S be all BFSs of L. Return $k \neq \emptyset$ if and only if $S \neq \emptyset$. It is correct by Claim 1, and we use the algorithm in Theorem 1 for runtime. □

2 Geometric View of LP

2.1 LPs as Shapes

Definition 3 (Polyhedron)**.** *A **polyhedron** is a set of the form*

$$K = \{x : Ax \leq b\}.$$

Definition 4 (Polytope)**.** *If there exists some $\Delta \in \mathbb{R}$ such that for any $x, y \in K$,*

$$d(x, y) \leq \Delta$$

*then K is a **polytope**.*

2.2 Inner Products

Consider the geometry of $\langle u, u \rangle$. From the definition of inner product, we know

$$\langle u, u \rangle \coloneqq \sum_i u_i^2.$$

Taking square root on both sides yields

$$\sqrt{\langle u, u \rangle} = \sqrt{\sum_i u_i^2} = d(0, u) = \|u\|,$$

which is the length of u.

Now, consider the geometry of $\langle u, v \rangle$ where

$$\langle u, v \rangle \coloneqq \sum_i u_i v_i.$$

Let

$$c^* \coloneqq \arg\min_c d(cv, u)$$

and let $\mathrm{Proj}(u \to v) = c^* v$. By calculus, we get $c^* = \frac{\langle u,v \rangle}{\langle v,v \rangle}$, so

$$\mathrm{Proj}(u \to v) = \langle u, v \rangle \frac{v}{\langle v, v \rangle}.$$

Hence, $\langle u, v \rangle$ describes "how much of u is in the direction of v."

Example 1. *When $\langle u, v \rangle = 0$, according to our interpretation, no component of u is in the direction of v, which means that u, v are orthogonal (or normal).*

(a) Projection of u onto v.

(b) Orthogonal case ($\langle u, v\rangle = 0$).

Figure 1: Geometric interpretation of the inner product and vector projection.

Definition 5 (Hyperplane, Normal Vector)**.** *Let $v \in \mathbb{R}^n$ and $c \in \mathbb{R}$. $\{u : \langle u, v\rangle = c\}$ is called a* ***hyperplane****. We just showed that*

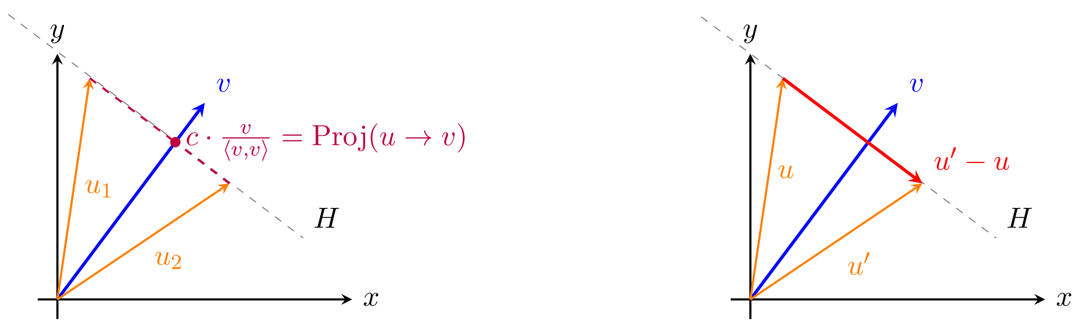
$$\langle u, v\rangle = c \Longleftrightarrow \mathrm{Proj}(u \to v) = c \cdot \frac{v}{\langle v, v\rangle}.$$

Hence, for any $u, u' \in$ hyperplane,

$$\mathrm{Proj}(u \to v) = \mathrm{Proj}(u' \to v).$$

v is called the ***normal vector****. v is "normal" in the sense that for any $u, u' \in$ hyperplane, $u \neq u'$,*

$$\langle u' - u, v\rangle = \langle u', v\rangle - \langle u, v\rangle = c - c = 0.$$



(a) Hyperplane H and constant projection onto normal vector v.

(b) Orthogonality of difference vector $u' - u$ and normal vector v.

Figure 2: Geometric properties of hyperplanes and their normal vectors.

Definition 6 (Halfspace)**.** *$\{u : \langle u, v\rangle \leq c\}$ is called a* ***halfspace****.*

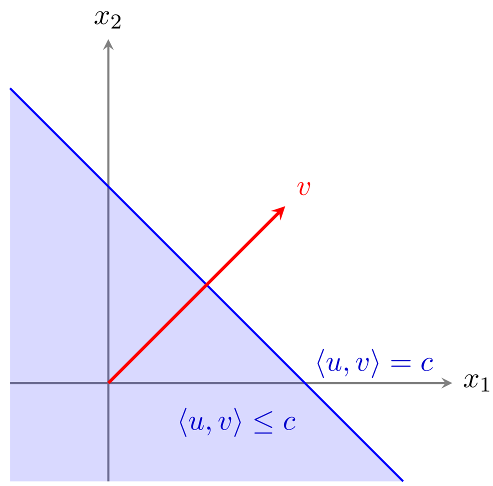


Figure 3: A halfspace defined by $\langle u, v\rangle \leq c$.

Remark 2. *A polyhedron is the intersection of m halfspaces, since a polydedron can be written as*

$$K = \{x : \langle a_i, x\rangle \leq b_i \quad \forall a_i \in \text{rows}(A)\}.$$

Notice that if x is tight for $\langle a_i, x\rangle \leq b_i$, then x is in the hyperplane $\{x : \langle a_i, x\rangle = b\}$.

2.3 Ellipsoid Algorithm

Key Takeaway: We can solve LP feasibility in Polynomial time (even with exponentially-many constraints).

We make the following assumptions:

(1) Given x_0, R such that $K \subseteq B(x_0, R)$.

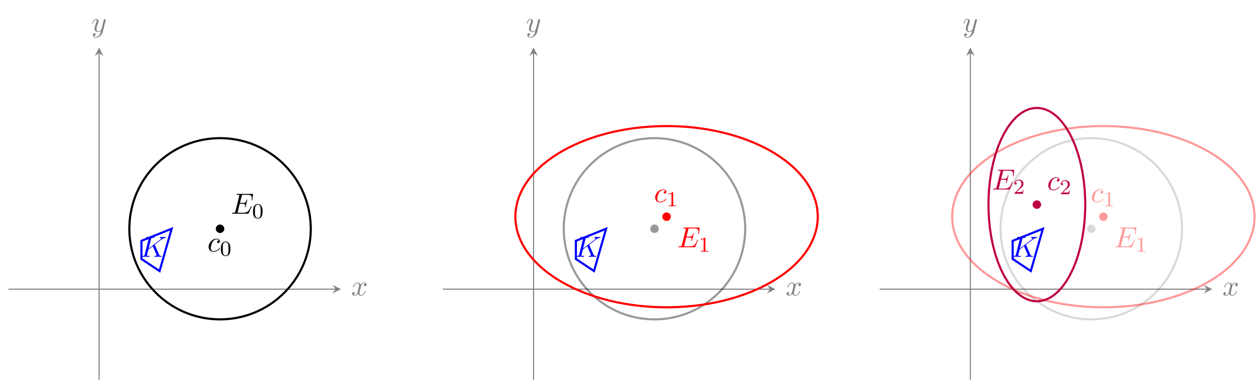
(2) There exist y_0 and $r < R$ such that if $K \neq \emptyset$, $B(y_0, R) \subseteq K$ with $\frac{R}{r} \leq \exp(\text{poly}(n))$.

(3) We have a "strong separation oracle" for K in $\text{poly}(n)$ times.

Given $y \in \mathbb{R}^n$, a *strong separation oracle* for $K = \{x : Ax \leq b\}$ either correctly outputs "$y \in K$" or returns an $a_j \in \text{rows}(A)$ such that $\langle y, a_j \rangle > b_j$.

Algorithm 2 Ellipsoid Algorithm

$E_0 \leftarrow B(x_0, R)$
for $i = 1, 2, \ldots, 10n^2 \cdot \ln(\frac{R}{r})$ **do**
    $c_{i-1} \leftarrow \text{center}(E_{i-1})$
    if $c_{i-1} \in K$ **then**
        return $K \neq \emptyset$ ▷ via separation oracle
    else
        Let $a_j \in \text{rows}(A)$ such that $\langle a_j, c_{i-1} \rangle > b_j$ ▷ via separation oracle
    end if
    $T_i \leftarrow E_{i-1} \cap \{x : \langle a_j, x \rangle \leq \langle a_j, c_{i-1} \rangle\}$
    Let E_i be a carefully chosen "ellipsoid" containing T_{i-1}
end for
return $K = \emptyset$



(a) Initial configuration. (b) Iteration Step 1. (c) Iteration Step 2.

Figure 4: Visualization of steps in the ellipsoid algorithm, showing iterative space refinement.

Lemma 4 (Key Lemma)**.** *We can compute E_i in* poly(n) *time such that*

$$T_i \subseteq E_i \quad \text{and} \quad \mathrm{vol}(E_i) \leq \left(1 - \frac{1}{5n}\right) \mathrm{vol}(E_{i-1})$$

Theorem 3. *Given the assumptions (1)-(3), the ellipsoid algorithm solves LP Feasibility in* poly(n) *time.*

Proof. Correctness:
If $K = \emptyset$ then the algorithm never returns $K \neq \emptyset$;
If $K \neq \emptyset$, by assumption, there exist r, y_0 as described, suppose for contradiction that the algorithm returned $K = \emptyset$.
Let $k = 10n^2 \cdot \ln(\frac{R}{r})$ be the number of iterations. By Key Lemma, $K \subseteq E_i$ for all i and $B(y_0, r) \subseteq K$, so $B(y_0, r) \subseteq E_k$. But $\mathrm{vol}(B(y_0, r)) = v_n \cdot r^n$, hence

$$\mathrm{vol}(E_k) \geq v_n \cdot r^n.$$

On the other hand, by Key Lemma and $\mathrm{vol}(E_0) = v_n \cdot R^n$, we have

$$\begin{aligned}
\mathrm{vol}(E_k) &\leq \left(1 - \frac{1}{5n}\right) \cdot \mathrm{vol}(E_{k-1}) \\
&\leq \left(1 - \frac{1}{5n}\right)^2 \cdot \mathrm{vol}(E_{k-2}) \leq \cdots \leq \left(1 - \frac{1}{5n}\right)^k \cdot \mathrm{vol}(E_0) \\
&= v_n \cdot R^n \cdot \left(1 - \frac{1}{5n}\right)^k \\
&\leq v_n \cdot R^n \cdot \exp\left(\frac{-k}{5n}\right) \\
&= v_n \cdot R^n \cdot \exp\left(-2n \cdot \ln\left(\frac{R}{r}\right)\right) \\
&= v_n \cdot R^n \cdot \left(\frac{r}{R}\right)^{2n} \\
&= v_n \cdot \left(\frac{r}{R}\right)^n \cdot r^n \\
&< v_n \cdot r^n
\end{aligned}$$

which contradicts that $\mathrm{vol}(E_k) \geq v_n \cdot r^n$.

Runtime:
The runtime of the algorithm is

$$T_{\text{separation}} \cdot O\left(n^2 \cdot \ln\left(\frac{R}{r}\right)\right) = \mathrm{poly}(n)$$

since $R/r \leq \exp(\mathrm{poly}(n))$. □