An Algorithmist's Toolkit — Brown CSCI 2952T – Spring 2026

Lecture 7: LPs—Duality, Hyperplane Separation, Ellipsoid

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1 Convexity

1.1 Intro to Convex Geometry

Definition 1 (Convexity)**.** *$w \in \mathbb{R}^n$ is a **convex combination** of $u, v \in \mathbb{R}^n$ iff $\exists p \in [0,1]$ s.t. $w \in [u, v]$ where*

$$[u, v] := \{p \cdot u + (1-p) \cdot v : p \in [0,1]\}$$

Definition 2 (Convex)**.** *$K \subseteq \mathbb{R}^n$ is **convex** iff*

$$\forall u, v \in K, [u, v] \subseteq K$$

Proposition 1. *Polyhedra $K = \{x : Ax \leq b\}$ is convex.*

Proof. If $u, v \in K$, then $Au \leq b$ and $Av \leq b$. Then, for any $p \in [0,1]$, we have

$$A(pu + (1-p)v) = pAu + (1-p)Av \leq pb + (1-p)b = b$$

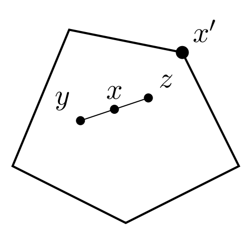
this essentially follows from the fact that A is linear and the fact that $\leq$ is preserved under convex combinations. □

1.2 Equivalence of Corners

We'll be talking about 3 different definitions of corners of a polyhedron K including BFS, which we've defined last class. We will show that these 3 different types of corners are actually the same thing.

Definition 3 (Extreme Point)**.** *x is an **extreme point** of K iff $x \in K$ and $x \in [y, z]$ for $y, z \in K$ implies $y = z$ (note, this also means that $x = y = z$.). I.e. there are no two distinct points $y, z \in K$ such that x is a convex combination of y and z.*

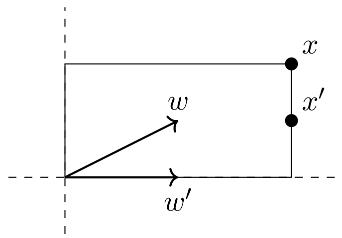
Example 1.



Here x' is an extreme point and x is a non example of an extreme point since x is a convex combination of y and z where $y \neq z$.

Definition 4 (Vertex)**.** *x is a **vertex** of K iff $x \in K$ and there exists a w s.t. $\langle x, w\rangle > \langle y, w\rangle$ for all $y \in K$ with $y \neq x$. I.e. there is a linear function that is uniquely maximized at x.*

Example 2.



Here x is a vertex since the linear function defined by w is uniquely maximized at x. However, a point x' is not a vertex since the linear function defined by w' is not uniquely *maximized at x'.*

Proposition 2 (Equivalence of Corners)**.** *$x \in K$, $K = x : Ax \leq b$ is a BFS iff x is a extreme point iff x is a vertex.*

Proof. We'll show how each definition implies the next cyclically to show equivalence:

BFS $\implies$ Vertex: Since $x \in K$ is a BFS, there exists a subset of tight constraints at x whose corresponding row vectors form a basis B of $\mathbb{R}^n$. That is, $B \subseteq \text{Tight}_A(x)$. Because the rows in B form a basis, A_B is invertible, meaning x is the unique point in $\mathbb{R}^n$ that satisfies $A_B x = b_B$.

To show x is a vertex, we must construct a vector w such that $\langle w, x\rangle > \langle w, y\rangle$ for all $y \in K$ with $y \neq x$. Let w be the sum of the normal vectors of our basis constraints:

$$w = \sum_{i \in B} a_i$$

First, let's evaluate the inner product of w with x. By linearity of the inner product, we have:

$$\begin{aligned}\langle w, x\rangle &= \left\langle \sum_{i \in B} a_i, x \right\rangle \\ &= \sum_{i \in B} \langle a_i, x\rangle \\ &= \sum_{i \in B} b_i\end{aligned}$$

where the last equality holds because x is tight for all constraints in B.

Now, take any other point $y \in K$ with $y \neq x$. Because y is in the polyhedron K, it must satisfy all constraints, meaning $\langle a_i, y\rangle \leq b_i$ for all $i \in B$. Therefore:

$$\langle w, y\rangle = \sum_{i \in B} \langle a_i, y\rangle \leq \sum_{i \in B} b_i = \langle w, x\rangle$$

For this to be an equality (i.e., $\langle w, y\rangle = \sum_{i\in B} b_i$), we would need $\langle a_i, y\rangle = b_i$ for *every* $i \in B$, since no individual term in the sum can exceed b_i.

However, we already established that x is the unique point satisfying $A_B x = b_B$. Because $y \neq x$, y cannot be tight for all constraints in B. It must be strictly less than b_i for at least one $i \in B$.

Therefore, the sum is strictly less, giving us the strict inequality:

$$\langle w, y\rangle < \sum_{i\in B} b_i = \langle w, x\rangle$$

This proves that x is uniquely maximized in the direction of w, so x is a vertex.

Vertex $\implies$ Extreme Point: By contrapositive, suppose x is not an extreme point. Then there exists $y, z \in K$ with $y \neq z$ such that $x \in [y, z]$. Then for any w, we have

$$\langle x, w\rangle = p \cdot \langle y, w\rangle + (1 - p) \cdot \langle z, w\rangle$$

for some $p \in [0, 1]$. So, either $\langle y, w\rangle \geq \langle x, w\rangle$ or $\langle z, w\rangle \geq \langle x, w\rangle$. This is because that $\langle x, w\rangle$ can be thought of an average between $\langle y, w\rangle$ and $\langle z, w\rangle$ - either one must be at least as large as the average.

In either case, x is not a vertex.

Extreme Point $\implies$ BFS: By contrapositive, suppose x is not a BFS. Then $\text{rank}(A^T) < n$ and there exists a nonzero $w \in \text{Ker}(A^T)$. From last class we have a lemma which says $\exists\epsilon$ s.t. $x \pm \epsilon w \in K$

$$x = \frac{1}{2}(x + \epsilon w) + \frac{1}{2}(x - \epsilon w)$$

Therefore x is not an extreme point.

□

1.3 Polytopes and Convex Hulls

Definition 5 (Convex Combination)**.** *w is a* ***convex combination*** *of $V \subseteq \mathbb{R}^n$ if*

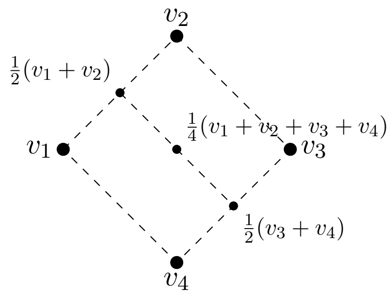
$$w = \sum_{v_i \in V} p_i v_i$$

where $p_i \geq 0$ for all i and $\sum_i p_i = 1$.

Definition 6 (Convex Hull)**.** *The* ***convex hull*** *of a set $V \subseteq \mathbb{R}^n$ is the set of all convex combinations of points in V.*

$$\text{Con}(V) = \left\{ \sum_{v_i \in V} p_i v_i : p_i \geq 0, \sum_i p_i = 1 \right\}$$

Proposition 3. *If K is a polytope with BFSs/vertices/extreme points V then $K = \text{Con}(V)$ i.e. every point in K can be written as a convex combination of the BFSs/vertices/extreme points of K.*

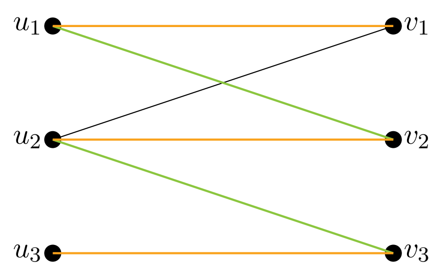


2 Integrality

2.1 Maximum Bipartite Matching

Definition 7 (Maximum Bipartite Matching (MBM))**.** *Given a graph $G = (V, E)$, find a matching $M \subseteq E$ of maximum size. I.e. find a set of edges M such that no two disjoint edges in M share an endpoint and $|M|$ is maximized.*

Example 3.



A bipartite graph with a maximum matching of size 3 (shown in red).

Definition 8 (IP Formulation of MBM)**.** *The MBM problem is equivalent to the following IP problem.*

Take the bipartite graph $G = (V, E)$ where $E = e_1, \cdots, e_n$ and let $M = |E|$. For $x \in \mathbb{R}^M$, notate x_i as x_{e_i} (indexing on the edges). For $u \in V$, let

$X(\delta(u)) \coloneqq \sum_{e \in \delta(u)} x_e$ where $\delta(u)$ is the set of edges incident to u.

our goal is to find $x \in \mathbb{R}^M$ maxing $\sum_{e \in E} x_e$ s.t. $X(\delta(u)) \leq 1$ for all $u \in V$ and $x_e \in \{0, 1\}$ for all $e \in E$.

This is a integer program and not a linear program because of the constraint $x_e \in \{0, 1\}$ for all $e \in E$. If w relax this constraint to $x_e \in [0, 1]$ for all $e \in E$ we have a linear program.

2.2 Interger Programs and Linear Programs

Suppose $K = x : Ax \leq b$ and $c \in \mathbb{R}^n$ and let:

$$\text{OPT}_{LP} \coloneqq \max_{x \in K} c^T x$$

$$\text{OPT}_{IP} \coloneqq \max_{x \in K, x \in \mathbb{Z}^n} c^T x$$

Finding $x \in k \cap \mathbb{Z}^n$ s.t.

$$\langle c, x \rangle = \text{OPT}_{IP}(c)$$

and finding $x \in K$ s.t.

$$\langle c, x \rangle = \text{OPT}_{LP}(c)$$

is an integer program and a linear program respectively. Note that $\text{OPT}_{LP} \supseteq \text{OPT}_{IP}$ since $K \supseteq K \cap \mathbb{Z}^n$.

Integer programs are NP-hard in general, but linear programs can be solved in polynomial time from what we've seen in class.

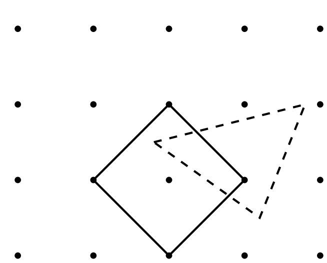
2.3 The LP Framework

Definition 9 (The LP Framework). *Since LPs are easier to solve than IPs, we can use the following framework to solve IPs:*

- *Express the problem as an integer program (IP)*
- *Get optimal solution to the linear program (LP) relaxation of the IP*
- *Use the solution to the LP to get a solution to the IP*

Definition 10 (Integral). *K is* ***integral*** *if all BFSs are integral. I.e. all BFSs have integer coordinates.*

Example 4.



Example (solid) and non example (dashed) of an integral polyhedron.

2.4 Back to Matching

We want to show the MBM is integral.

Proposition 4. *The maximum bipartite matching polytope*

$$K_M = \{x : x(\delta(u)) \le 1 \ \forall u \in V, \ \ x_e \in [0,1]\}$$

is integral.

Proof. Let y be a BFS of $K_M \leftrightarrow y$ is an extreme point. We show y must be integral.

Let

$$F = \{e \in E : y_e \in (0,1)\}$$

be the set of fractional edges, and consider the graph (V, F).

If $F = \emptyset$ we are done. Assume for the sake of contradiction that $|F| \ge 1$. Each vertex incident to a fractional edge must be tight, i.e. they must satisfy

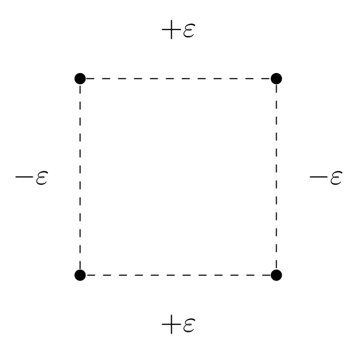
$$y(\delta(u)) = 1.$$

Otherwise, we could perturb the edge value by a small ε without violating feasibility.

Thus every vertex in (V, F) has degree at least 2, so the graph contains either a path or a cycle.

Cycle case

Since the graph is bipartite, every cycle has even length.



Assign alternating perturbations $+\varepsilon, -\varepsilon$ along the cycle. Because the cycle length is even, the sum of incident edges at each vertex remains unchanged.

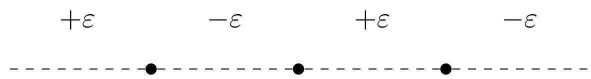
Thus both $y + \varepsilon d$ and $y - \varepsilon d$ remain feasible, where d is the alternating direction vector. Hence

$$y = \tfrac{1}{2}(y + \varepsilon d) + \tfrac{1}{2}(y - \varepsilon d)$$

so y is not an extreme point.

Path case

If (V, F) contains a path, we again alternate perturbations along the edges.



Choosing ε sufficiently small preserves feasibility. Again we obtain two distinct feasible points $y \pm \varepsilon d$ whose average is y, contradicting extremality. Note that the outer edges of the path are not tight, which is why we can we can perturb the outer edges.

Thus no BFS can contain fractional edges, so every BFS is integral. □

3 Duality

3.1 Duality: The Study of Infeasibility

Suppose we want to check feasibility of the system

$$Ax \leq b$$

i.e.

$$\begin{aligned} \langle a_1, x \rangle &\leq b_1 \\ \langle a_2, x \rangle &\leq b_2 \\ &\vdots \\ \langle a_m, x \rangle &\leq b_m. \end{aligned}$$

We ask whether there exists x satisfying all inequalities.

Observe that if we take any nonnegative vector

$$\lambda \in \mathbb{R}^m_{\geq 0}$$

then multiplying each inequality by λ_i and summing gives

$$\langle A^T \lambda, x \rangle \leq \langle \lambda, b \rangle.$$

3.2 Farkas' Lemma

Proposition 5 (Farkas' Lemma)**.**

$$\exists \lambda \geq 0 : A^T \lambda = 0,\ \langle \lambda, b \rangle < 0 \quad \iff \quad \nexists x \ s.t. \ Ax \leq b.$$

Proof. (forward direction). Suppose for contradiction that there exists x with $Ax \le b$. Since $\lambda \ge 0$, multiplying each inequality by λ_i and summing gives

$$\langle \lambda, Ax \rangle \le \langle \lambda, b \rangle.$$

Using the identity $\langle \lambda, Ax \rangle = \langle A^T \lambda, x \rangle$, we obtain

$$\langle A^T \lambda, x \rangle \le \langle \lambda, b \rangle.$$

But $A^T \lambda = 0$, so

$$0 = \langle A^T \lambda, x \rangle \le \langle \lambda, b \rangle.$$

This contradicts the assumption that $\langle \lambda, b \rangle < 0$.
Therefore no x satisfying $Ax \le b$ can exist. □

3.3 Optimization Duality

Consider the linear program

$$\text{(Primal)} \qquad \max \langle c, x \rangle \quad \text{s.t. } Ax \le b.$$

Let

$$P := \max_{x : Ax \le b} \langle c, x \rangle.$$

Suppose there exists $\lambda \ge 0$ such that

$$A^T \lambda = c.$$

Then for any feasible x,

$$\langle c, x \rangle = \langle A^T \lambda, x \rangle = \langle \lambda, Ax \rangle \le \langle \lambda, b \rangle.$$

Thus

$$P \le \langle \lambda, b \rangle.$$

So every such λ provides an upper bound on the optimal value.

To obtain the tightest bound, we minimize over all such λ.

$$\text{(Dual)} \qquad D = \min_{\lambda \ge 0} \langle \lambda, b \rangle \quad \text{s.t. } A^T \lambda = c.$$

3.4 Primal–Dual Pair

$$\textbf{Primal:} \qquad \max\langle c, x\rangle \quad \text{s.t. } Ax \le b$$

$$\textbf{Dual:} \qquad \min\langle \lambda, b\rangle \quad \text{s.t. } A^T\lambda = c, \quad \lambda \ge 0$$

Every dual feasible solution gives an upper bound on the primal objective.

$$\langle c, x\rangle \le \langle \lambda, b\rangle.$$

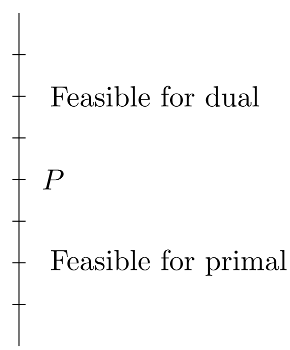
3.5 Weak Duality

Proposition 6 (Weak Duality)**.** *If x is primal feasible and λ is dual feasible then*

$$\langle c, x\rangle \le \langle \lambda, b\rangle.$$

or in other words,

$$P \le D$$



3.6 Strong Duality

Proposition 7 (Strong Duality)**.** *If the primal LP has an optimal solution then*

$$P = D.$$

Thus the optimal value of the primal equals the optimal value of the dual.

3.7 Primal–Dual Correspondence

Primal	**Dual**
max	min
objective	bound
$\langle x, c \rangle$	$A^T \lambda = c$
bound	objective
$Ax \leq b$	$\langle \lambda, b \rangle$
constraint	variable
$Ax \leq b$	λ_i
variable	constraint
x_i	$\sum_{i=1}^{m} A_{ij} \lambda_i = c_j \ \forall j$