Lecture 5: LPs—the Algebraic and Linear Algebraic Views

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1 LP Feasibility

We begin with the linear programming feasibility problem.

Definition 1 (LP Feasibility)**.** *Given a matrix $A \in \mathbb{R}^{m\times n}$ and a vector $b \in \mathbb{R}^m$, decide whether there exists $x \in \mathbb{R}^n$ such that*

$$Ax \le b.$$

Equivalently, determine whether the polyhedron

$$K := \{x \in \mathbb{R}^n : Ax \le b\}$$

is nonempty.

Example (Resource Allocation). Suppose we wish to determine whether we can grow 10 lbs of strawberries and 5 lbs of watermelon given resource constraints. Let

$$x_1 = \text{lbs of strawberries}, \qquad x_2 = \text{lbs of watermelon}.$$

The constraints can be written as

$$\begin{aligned} -x_1 &\le -10, \\ -x_2 &\le -5, \\ 5x_1 + 25x_2 &\le 200, \\ 20x_1 + 5x_2 &\le 100. \end{aligned}$$

In matrix form:

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 5 & 25 \\ 20 & 5 \end{pmatrix}, \quad b = \begin{pmatrix} -10 \\ -5 \\ 200 \\ 100 \end{pmatrix}.$$

The feasibility problem asks whether there exists $x \in \mathbb{R}^2$ such that $Ax \le b$.

1.1 Algebra of Inequalities

We describe operations that preserve feasibility.

For any $x \in \mathbb{R}^n$:

1. For any $c \in \mathbb{R}$,

$$\langle a, x\rangle \le b \iff \langle a, x\rangle + c \le b + c.$$

2. For any $\lambda > 0$,
$$\langle a, x\rangle \le b \quad \Longleftrightarrow \quad \lambda\langle a, x\rangle \le \lambda b.$$

3. $$\langle a, x\rangle \le b \quad \Longleftrightarrow \quad -\langle a, x\rangle \ge -b.$$

These transformations preserve feasibility of the system.

1.2 Fourier–Motzkin Elimination

We now describe an algorithm for eliminating variables from systems of linear inequalities.

Definition 2 (Isolated Variable)**.** *We say that a variable x_i is* isolated *in an inequality if the inequality is of the form*
$$\ell \le x_i \quad \textit{or} \quad x_i \le r.$$

Fourier–Motzkin Elimination. For $i = 1, 2, \ldots, n$:

1. Isolate x_i in all inequalities containing x_i.
2. For each pair of inequalities of the form
$$\ell \le x_i, \qquad x_i \le r,$$
add the inequality $\ell \le r$.
3. Delete all inequalities containing x_i.

The resulting system no longer contains x_i. The original system is feasible if and only if the final reduced system is feasible.

Example (Fourier–Motzkin). Consider the system:

$$\begin{aligned} -x_1 &\le -10 \\ -x_2 &\le -5 \\ 5x_1 + 25x_2 &\le 200 \\ 20x_1 + 5x_2 &\le 100 \end{aligned}$$

Step 1: Isolate x_1. Rewrite inequalities involving x_1:

$$-x_1 \le -10 \quad \Longrightarrow \quad 10 \le x_1$$

$$5x_1 + 25x_2 \le 200 \quad \Longrightarrow \quad x_1 \le 40 - 5x_2$$

$$20x_1 + 5x_2 \le 100 \quad \Longrightarrow \quad x_1 \le 5 - \tfrac{1}{4}x_2$$

So we obtain bounds of the form:

$$10 \le x_1$$

$$x_1 \le 40 - 5x_2$$

$$x_1 \le 5 - \tfrac{1}{4}x_2$$

Step 2: Combine lower and upper bounds. From $10 \le x_1 \le 40 - 5x_2$, we get:

$$10 \le 40 - 5x_2 \quad \Longrightarrow \quad x_2 \le 6$$

From $10 \le x_1 \le 5 - \frac{1}{4}x_2$, we get:

$$10 \le 5 - \tfrac{1}{4}x_2 \quad \Longrightarrow \quad x_2 \le -20$$

We keep the remaining inequality:

$$-x_2 \le -5 \quad \Longrightarrow \quad 5 \le x_2$$

Step 3: Final reduced system in x_2.

$$5 \le x_2$$

$$x_2 \le 6$$

$$x_2 \le -20$$

This system is infeasible, since $x_2 \le -20$ contradicts $x_2 \ge 5$.

Hence the original LP is infeasible.

1.3 Correctness of Fourier–Motzkin

Lemma 1. *Eliminating a variable via Fourier–Motzkin preserves feasibility.*

Proof Sketch. Suppose x satisfies all inequalities before eliminating x_i. Then for each pair $\ell \le x_i$ and $x_i \le r$, we have

$$\ell(x) \le x_i \le r(x),$$

hence $\ell(x) \le r(x)$. Thus x satisfies all newly generated inequalities.

Conversely, if all inequalities after elimination are satisfied, then there exists a value c such that

$$\ell(x) \le c \le r(x)$$

for all lower and upper bounds. Setting $x_i = c$ yields a feasible solution to the original system. □

1.4 Runtime of Fourier–Motzkin

Let m be the number of inequalities and n the number of variables.

Suppose that when eliminating a variable x_i:

- There are m^+ inequalities giving upper bounds $x_i \leq r$,
- There are m^- inequalities giving lower bounds $\ell \leq x_i$.

When eliminating x_i, we generate at most

$$m^+ \cdot m^-$$

new inequalities by pairing each lower bound with each upper bound.

In the worst case,

$$m^+ \leq m, \quad m^- \leq m,$$

so up to m^2 inequalities may be produced.

Growth Across Eliminations. After eliminating one variable:

$$m_1 \leq m^2.$$

After eliminating two variables:

$$m_2 \leq (m^2)^2 = m^4.$$

Continuing this process, after eliminating n variables:

$$m_n \leq m^{2^n}.$$

Thus, the total runtime is bounded by

$$O(m^{2^n}),$$

which is exponential in the number of variables n.

2 LP Search Problem

Given $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$, we want to either

1. output x such that $Ax \leq b$, or
2. report no such x

Notation: let $N = m + n$ from now on.

Theorem 1. *If we can solve LP feasibility in $T(N)$ time, then we can solve LP search in* $\text{poly}(N) \cdot T(N)$ *time.*

To prove this theorem, we will utilize three lemmas.

Definition 3. *An LP is in equational form if we have the constraints $Ax = b, x \geq 0$*

Lemma 2. *If we can solve equational form LP in $T(N)$ time, then we can solve LP search in $T(O(N)))$ time.*

Proof. Note that

$$\exists x \in \mathbb{R}^n \text{ s.t } Ax \leq b \iff \exists x^+, x^- \in \mathbb{R}^n \text{ s.t } A(x^+ - x^-) \leq b, x^-, x^+ \geq 0$$

where we define x^+ as all the coordinates in x that are greater or equal to 0 (and setting every other coordinate to 0), and similarly x^- as the negative of all the coordinates in x that are less than or equal to 0. That is, $x = x^+ - x^-$.

The above equation is equivalent to

$$\exists x^+, x^-, s \in \mathbb{R}^n \text{ s.t } Ax^+ - Ax^- + s \leq b, x^-, x^+, s \geq 0 \tag{1}$$

where $s = b - A(x^+ - x^-)$. All if and only ifs should be trivial to see.

Putting Equation 1 in the equational form, we have

$$\begin{pmatrix} \vdots & \vdots & \vdots \\ A & -A & I \\ \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} x^+ \\ x^- \\ s \end{pmatrix} = b \tag{2}$$

as we can let $A' = \begin{pmatrix} \vdots & \vdots & \vdots \\ A & -A & I \\ \vdots & \vdots & \vdots \end{pmatrix}$ and $x' = \begin{pmatrix} x^+ \\ x^- \\ s \end{pmatrix} \geq 0$. Now we have m constants and $2n + m$ variables. Since$2n + m + m = O(N)$, we can solve LP search in $T(O(N))$ if we can solve the equational form in $T(N)$. ◻

Next, we have the following fact from Gaussian Elimination.

Fact: (Gaussian Elimination) Given $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$, let $K' = \{x \ : \ Ax = 0\}$. We can output $x \in K'$ or report than $K' = \varnothing$ in poly(N) time.

However, this alone is not enough to prove the theorem, as we want to look for nonnegative solutions. We will need another lemma.

Let $O_+ = \{x : \ x \geq 0\} \in \mathbb{R}^n$. Given, $S \subseteq [n]$, we let

$$K_S := \{x \mid Ax = b \text{ and } x_i = 0 \ \forall i \in S\}$$

Lemma 3. *$\forall S \subseteq [n]$, if $K_S \cap O_+ \neq \varnothing$, and $K_s \backslash O_+ \neq \varnothing$, then $\exists i \notin S$ such that $K_{S \cup \{i\}} \cap O_+ \neq \varnothing$*

Proof. Let

$$x \in K_s \cap O_+ \qquad y \in K_s \backslash O_+$$

For $p \in [0, 1]$, let

$$z_p := px + (1-p)y$$

such that $z_1 = x$ and $z_0 = y$. There exists some $i \notin S$ such that $x_i > 0$ and $y_i < 0$, so there exists some p such that $(z_p)_i = px_i + (1-p)y_i = 0$.

We let i be the largest p such that a sign changed in the coordinate compared to x. Then, $z_p \geq 0$. Moreover,

$$A(z_p) = A(p \cdot x + (1-p) \cdot y) = pAx + (1-p)Ay = pb + (1-p)b = b$$

so $\exists i \notin S$ and $z_p \in K_{S \cup i} \cap O_+$. □

Lemma 4. *If we can solve equational form LP feasibility in $T(N)$ time, then we can solve equational LP search in $\mathrm{poly}(N) \cdot T(O(N))$ time*

Proof. Below is the algorithm:

Algorithm 1 Finding solution to LP search

$S \leftarrow \varnothing$
for $n+1$ times **do**
    compute $y \in K_S$ (by Gaussian Elimination)
    if $y \in O_+$ **then**
        return y
    end if
    if $\exists i \notin S$ such that $K_{S+i} \cap O_+ \neq \varnothing$ **then**
        $S \leftarrow S + i$
    else
        return report $K = \varnothing$
    end if
end for

Correctness: If y is returned, then $Ay = b$ and $y \geq 0$. So we want to show that if $K := \{x \ : \ Ax = b, x \geq 0\} \neq \varnothing$, we return *something*. If $K \neq 0$, then $K_S \cap O_+ \neq \varnothing$. By induction, we always have $K_S \cap O_+ \neq \varnothing$, so the algorithm will either return y, or return 0 after iterating $n+1$ times, which is feasible.

Runtime: Each feasibility process has $\leq m+n$ constraints, of which we solve for a total $\leq n^2$ times, for a total of $n^2 T(O(N))$ runtime.

Each Gausian Elimination process has at most n variables and at most $m+n$ constraints, so the whole thing is $\mathrm{poly}(N)$ times running $n+1$ times, still $\mathrm{poly}(N)$ time. □

Lastly, we just need to piece together the lemma from Feasibility $\to$ Equational LP Search $\to$ LP Search to prove the theorem.

3 LP Optimization

Given A, b, and $c \in \mathbb{R}^n$, we want to either

1. output x such that $\langle c, x\rangle$ is max given $Ax \leq b$, or
2. report no such x

Theorem 2. *If we can solve LP search in $T(N)$ time, then we can solve LP optimization in* $\text{poly}(N) \cdot T(N)$.

Proof. We let

$$K_g = \{x \mid Ax \leq b \text{ and } \langle c, x\rangle \geq g\}$$

for some $g \in \mathbb{R}$. Denote the optimal solution by OPT. Assume we have L, R such that $L \leq OPT \leq R$, $|L - R| \leq \exp(\text{poly}(N))$ and $OPT \in \mathbb{Z}$. We can binary search OPT.

(The reason why we can assume $OPT \in \mathbb{Z}$ is because by some number theory trick whatever we can extend it without loss of generality to everything :) $\square$