

Lecture 10: l_1 , the Cut Metric Cone, Sparsest Cut, Expanders

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1 The l_1 Metric and the Cut Cone

Recall the following definitions and claims from last week's lectures.

Definition 1. A **metric space**, or “metric” for short, consists of a set of “points” V and function $f : V \times V \rightarrow \mathbb{R}_{\geq 0}$ satisfying

1. $f(x, x) = 0$ for all $x \in V$,¹
2. $f(x, y) = f(y, x)$ for all $x, y \in V$ (symmetry),
3. $f(x, y) \leq f(x, z) + f(z, y)$ for all $x, y, z \in V$ (triangle inequality).

Definition 2. An **embedding** of a metric space (V, d) into a metric space (V', d') is a function $f : V \rightarrow V'$.

Definition 3. A metric (V, d) **embeds isometrically** into a metric (V', d') iff there exists an embedding $f : V \rightarrow V'$ s.t. $d(u, v) = d'(f(u), f(v))$ for all $u, v \in V$.

Algorithm 1 Bourgain's Embedding**Require:** $x \in V$

- 1: Let $c \geq 1$ be a sufficiently large constant (chosen later)
- 2: **for** $j \in [\log n]$ **do** ▷ $|B|$ possibilities
- 3: **for** $i \in [c \log n]$ **do** ▷ repetitions
- 4: Construct S_{ij} by including each $v \in V$ independently with probability $\frac{1}{2^j}$
- 5: $\tilde{x}_{ij} \leftarrow \delta(x, S_{ij})$
- 6: **end for**
- 7: **end for**
- 8: **return** \tilde{x} ▷ $\tilde{x} \in \mathbb{R}^{c \log^2 n}$

Theorem 1 (Bourgain's). *Given any n -point metric (V, δ) , there exists a poly-time computable embedding $f : V \rightarrow V'$ with distortion $O(\log n)$ of (V, δ) into (\tilde{V}, d) , where d is the Euclidean distance, for some $\tilde{V} \subseteq \mathbb{R}^{O(\log^2 n)}$.*

Remark 1. *The $O(\log^2 n)$ above can be reduced to $O(\log n)$ with the Johnson-Lindenstrauss Lemma.*

Claim 1 (Expansion). *Given any n -point metric (V, δ) and the metric (\tilde{V}, d) given by Bourgain's Embedding for $c \geq 1$, for all $x, y \in V$ except with probability $\leq \frac{1}{n}$,*

$$\sum_{i,j} |\tilde{x}_{ij} - \tilde{y}_{ij}| \geq \frac{c}{40} \log n \cdot \delta(x, y).$$

¹Strictly, this defines a “pseudo metric,” as the metric definition would require $f(x, y) = 0$ iff $x = y$ for all $x, y \in V$.

1.1 ℓ_1 Metrics

So far, we have worked with “straight-line” distances in \mathbb{R}^n . Today, we study ℓ_1 distances in \mathbb{R}^n and their connection to cuts.

Definition 4. The function $d_1 : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ is defined for any $x, y \in \mathbb{R}^n$ as

$$d_1(x, y) \doteq \sum_i |x_i - y_i|.$$

See Figure 1 for an illustrative example contrasting d_1 with Euclidean distance.

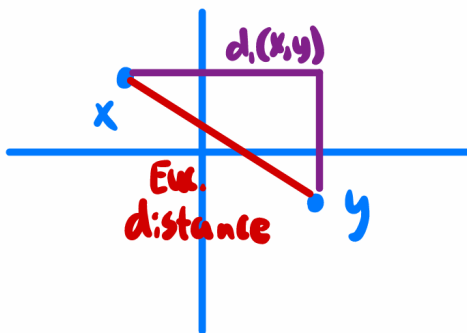


Figure 1: ℓ_1 .

Claim 2. (V, d_1) is a metric space for all (not necessarily finite) $V \subseteq \mathbb{R}^n$.

Proof. We have

1. $d_1(x, x) = \sum_i |x_i - x_i| = 0$ for all $x \in V$,
2. $d_1(x, y) = \sum_i |x_i - y_i| = \sum_i |y_i - x_i| = d_1(y, x)$ for all $x, y \in V$,
3. $d_1(x, y) = \sum_i |x_i - y_i| = \sum_i |x_i - z_i + z_i - y_i| \leq \sum_i |x_i - z_i| + \sum_i |z_i - y_i| = d_1(x, z) + d_1(z, y)$
for all $x, y, z \in V$ by the triangle inequality in \mathbb{R}^n , i.e., $|a + b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}^n$,

so (V, d_1) is indeed a metric space. □

Theorem 2 (Bourgain’s ℓ_1). *Given any n -point metric (V, δ) , there exists a poly-time computable embedding $f : V \rightarrow V'$ with distortion $O(\log n)$ of (V, δ) into (\tilde{V}, d_1) for some $\tilde{V} \subseteq \mathbb{R}^{O(\log^2 n)}$.*

Proof. Let $\hat{x} \doteq \frac{\tilde{x}}{\log n}$, where the \tilde{x} comes from Bourgain’s ℓ_2 embedding. We want to show that

$$\delta(x, y) \leq d_1(\hat{x}, \hat{y}) \leq O(\log n) \cdot \delta(x, y).$$

Starting with the second inequality,

$$\begin{aligned}
 d_1(\hat{x}, \hat{y}) &= \sum_{i,j} |\hat{x}_{ij} - \hat{y}_{ij}| \\
 &= \frac{1}{\log n} \sum_{i,j} |\delta(x, S_{ij}) - \delta(y, S_{ij})| && \text{(definition of } \hat{x} \text{ and } \hat{y}) \\
 &\leq \frac{1}{\log n} \sum_{i,j} \delta(x, y) && \text{(set triangle inequality)} \\
 &\leq c \log n \delta(x, y).
 \end{aligned}$$

Continuing with the first inequality,

$$\begin{aligned}
 d_1(\hat{x}, \hat{y}) &= \frac{1}{\log n} d_1(\tilde{x}, \tilde{y}) && \text{(definition of } \hat{x}) \\
 &\geq \frac{1}{\log n} \frac{c}{40} \log n \cdot \delta(x, y) && \text{(expansion claim)} \\
 &\geq \delta(x, y). && \text{(} c \text{ large)}
 \end{aligned}$$

Plugging this all together, we get

$$\delta(x, y) \leq d_1(\hat{x}, \hat{y}) \leq O(\log n) \cdot \delta(x, y),$$

as desired. □

1.2 ℓ_1 vs ℓ_2

We now generalize the d_1 defined on \mathbb{R}^n in the previous subsection to any space.

Definition 5. (V, δ) is an ℓ_1 **metric** (resp. ℓ_2 **metric**) if it embeds isometrically via $f : V \rightarrow V'$ into (\tilde{V}, d_1) (resp. (\tilde{V}, d) with d Euclidean), where $\tilde{V} \subseteq \mathbb{R}^n$ for some n .

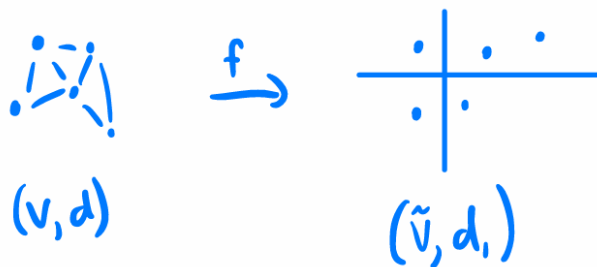


Figure 2: ℓ_1 metric.

See Figure 2 for an example of such f . For an arbitrary set of points V with $|V| = n$, let $\ell_1(V) \subseteq \mathbb{R}^{\binom{n}{2}}$ (resp. $\ell_2(V) \subseteq \mathbb{R}^{\binom{n}{2}}$) be all ℓ_1 (resp. ℓ_2) metrics on V .

Fact 1. For all V , we have $\ell_2(V) \subseteq \ell_1(V)$.

This, provable by using normally distributed random vectors to embed a Euclidean metric into an ℓ_1 space, gives ℓ_1 more representation power than ℓ_2 , as the inverse does not necessarily hold.

Fact 2. *There exists a V such that $\ell_1(V) \subsetneq \ell_2(V)$.*

Though what *exactly* can we represent with this additional power? The short answer is, cuts, and in fact, cuts provide a complete description of all ℓ_1 metrics.

1.3 Metrics as Vectors

Before showing that cuts provide a complete description of ℓ_1 metrics, we need some build-up.

Claim 3. *If (V, d) is a metric, then $(V, \alpha d)$ is a metric for all $\alpha \geq 0$.*

Proof. We can show all three properties of a metric,

1. $d(x, x) = 0$ for all $x \in V \implies \alpha d(x, x) = 0$ for all $x \in V$,
2. $d(x, y) = d(y, x)$ for all $x, y \in V \implies \alpha d(x, y) = \alpha d(y, x)$ for all $x, y \in V$,
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in V \implies \alpha d(x, y) \leq \alpha(d(x, z) + d(z, y)) = \alpha d(x, z) + \alpha d(z, y)$ for all $x, y, z \in V$.

□

Claim 4. *If (V, d_1) and (V, d_2) are metrics, then so is $(V, d_1 + d_2)$.*

Proof. Analogously to the implications above,

1. $d_1(x, x) + d_2(x, x) = 0$ for all $x \in V$,
2. $d_1(x, y) + d_2(x, y) = d_1(y, x) + d_2(y, x)$ for all $x, y \in V$,
3. $d_1(x, y) + d_2(x, y) \leq d_1(x, z) + d_2(x, z) + d_1(z, y) + d_2(z, y)$ for all $x, y, z \in V$ by the triangle inequality holding for both (V, d_1) and (V, d_2) .

□

The “life lesson” is that if you can add and scale certain mathematical objects, it may be good to interpret them as vectors. Indeed, there is a 1-1 **correspondence between metrics and vectors**, see Figure 3 for an example with the graph distance metric,

$$(V, d) \text{ s.t. } V = \{v_1, \dots, v_n\} \iff \begin{pmatrix} d(v_1, v_2) \\ d(v_1, v_3) \\ \dots \\ d(v_1, v_n) \\ d(v_2, v_1) \\ \dots \end{pmatrix} \in \mathbb{R}^{\binom{n}{2}}$$

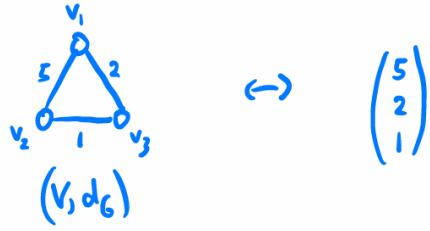


Figure 3: 1-1 correspondence between metrics and vectors.

Notation abuse. Use $d \in \mathbb{R}^{\binom{n}{2}}$ for vector of (V, d) , and call d a metric on V .

Definition 6. The *convex cone* of $D \subseteq \mathbb{R}^{\binom{n}{2}}$ is defined as

$$\text{Cone}(D) \doteq \left\{ \sum_{d \in D} \alpha_d d : \alpha_d \geq 0 \forall d \right\}.$$

For a minimal $D = \{d_1, d_2\}$, we have $[d_1, d_2] \in \text{Cone}(D)$, and so $x \in \text{Cone}(D)$ implies that $\alpha x \in \text{Cone}(D)$ for all $\alpha \geq 0$. See Figure 4 for an illustration.

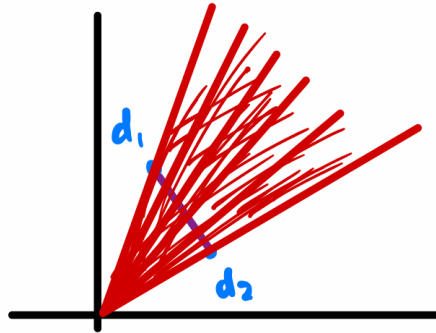


Figure 4: Convex cone.

Corollary 1. If D is a set of metrics on V , then $\tilde{d} \in \text{Cone}(D)$ is a metric on V .

Proof. This follows from Claims 3 and 4. □

Observation 1. If $d_1, d_2 \in \text{Cone}(D)$, then $d_1 + d_2 \in \text{Cone}(D)$.

Proof. Given any $d_1, d_2 \in \text{Cone}(D)$, we have

$$\begin{aligned} d_1 + d_2 &= \sum_{d \in D} \alpha_d d + \sum_{d \in D} \beta_d d \\ &= \sum_{d \in D} (\alpha_d + \beta_d) d, \end{aligned}$$

where $(\alpha_d + \beta_d) \geq 0$ for all $d \in D$, and so $d_1 + d_2 \in \text{Cone}(D)$. □

1.4 The Cut Cone

Now comes the main actor in our description of ℓ_1 metrics.

Definition 7. (V, d_S) is a **cut metric** for $S \subseteq V$ such that

$$d_S(u, v) = \begin{cases} 0 & u, v \in S \text{ or } u, v \notin S \\ 1 & \text{otherwise.} \end{cases}$$

See this definition illustrated in Figure 5.

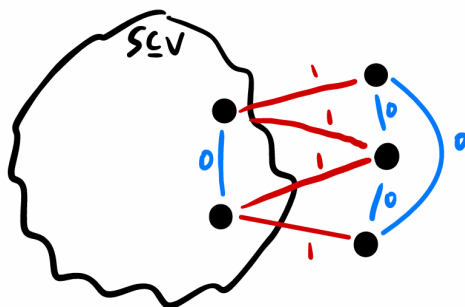


Figure 5: Cut metric

The cut metric is a metric because (1) and (2) clearly hold, and (3) is illustrated in Figure 6. If two points are on opposite sides of the boundary of S , then a third point is on the opposite side of the “boundary” with respect to exactly one of the other two.

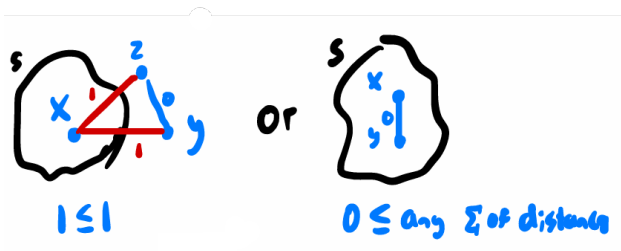


Figure 6: Triangle inequality explanation.

For an arbitrary $V \subseteq \mathbb{R}^k$, where $|V| = n$, let $\text{Cut}(V) \subseteq \mathbb{R}^{\binom{n}{2}}$ be all cut metrics of V .

Theorem 3. For all V , $\ell_1(V) = \text{Cone}(\text{Cut})(V)$.

Also, given $d \in \ell_1(V)$ and ℓ_1 embedding, we can poly-time compute $d_i \in \text{Cut}(V)$, $\alpha_i \in \mathbb{R}$ for all i such that $d = \sum_i \alpha_i d_i$. We start with the simplest case of line metrics.

Definition 8. (V, d) is a **line metric** if it embeds isometrically into (\tilde{V}, \tilde{d}) for $\tilde{V} \subseteq \mathbb{R}$ and $\tilde{d}(x, y) = |x - y|$ for all $x, y \in \tilde{V}$.

See Figure 7 for an illustration of a line metric.



Figure 7: Line metric.

Claim 5. *If d is a line metric on V , then $d \in \text{Cone}(\text{Cut}(V))$.*

Also, we can poly-time compute $d_i \in \text{Cut}(V)$, $\alpha_i \in \mathbb{R}$ for all i such that $d = \sum_i \alpha_i d_i$. By notation abuse, we use $v \in V$ and the position on the real line interchangeably. Let $V = \{v_1, \dots, v_n\}$ be sorted from the left to the right. Let $S_i \doteq \{v_1, \dots, v_i\}$, and let d_{S_i} be the corresponding cut metric for each $i < n$. See Figure 8 for an illustration of this sorting and metrization.

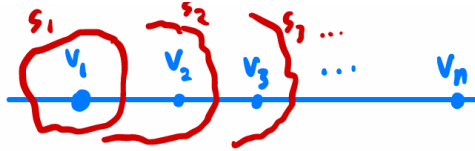


Figure 8: All stuff on one or the other side of the “boundary” of S_i .

Let $\alpha_i \doteq v_{i+1} - v_i$, which is ≥ 0 by definition. Then, we have $d = \sum_i \alpha_i d_{S_i}$ because for all $y \geq x$,

$$\begin{aligned}
 \sum_i \alpha_i d_{S_i}(v_x, v_y) &= \sum_{i: S_i \text{ separates } v_x, v_y} (v_{i+1} - v_i) \\
 &= \sum_{i \in [x, y)} (v_{i+1} - v_i) \\
 &= v_y - v_x && \text{(telescoping)} \\
 &= d(v_x, v_y).
 \end{aligned}$$

So, $d \in \text{Cone}(\text{Cut}(V))$. Poly-time computation trivial. We now finally prove Theorem 3.

Proof. We need to show both subset inclusions.

$\ell_1(V) \subseteq \text{Cone}(\text{Cut}(V))$. If d is an ℓ_1 metric, then $d = \sum_i d_i$, where each d_i is a line metric, see Figure 9. But each $d_i \in \text{Cone}(\text{Cut}(V))$ by Claim 5, so by Observation 1, we have $\sum_i d_i \in \text{Cone}(\text{Cut}(V))$. Poly-time computation is trivial assuming that d is embedded into ℓ_1 with polynomially many dimensions.

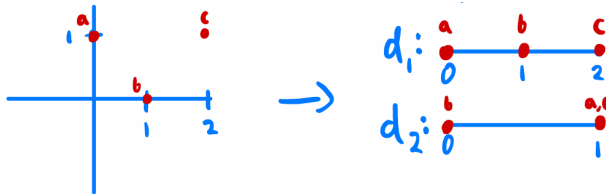


Figure 9: An ℓ_1 metric is a line metric.

$\text{Cone}(\text{Cut}(V)) \subseteq \ell_1(V)$. Observe that if d is a cut metric, then it is also a line metric, see Figure 10.

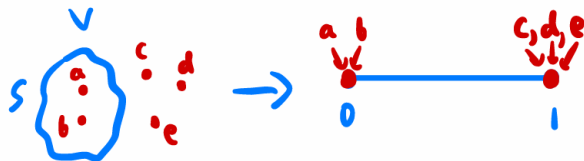


Figure 10: A cut metric is a line metric.

Also, if d is a line metric, then so is αd for any $\alpha \geq 0$, see Figure 11.



Figure 11: Line metrics closed under scaling.

Finally, $\sum_i d_i$ for d_i a line metric is an ℓ_1 metric, see Figure 12.

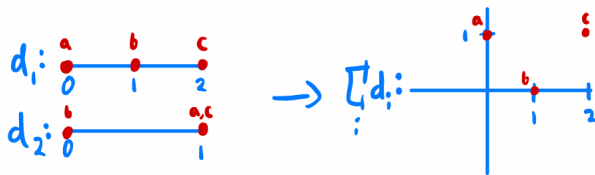


Figure 12: A line metric is an ℓ_1 metric.

So, $d \in \text{Cone}(\text{Cut}(V)) = \sum_i \alpha d_i$. By d_i being a cut and thus line metric, αd_i is also a line metric, and so $\sum_i \alpha d_i$ is an ℓ_1 metric. Therefore, $d \in \ell_1(V)$ as desired. \square

2 Sparsest Cut and Expanders

2.1 Sparsest Cut

Definition 9. Given connected graph $G = (V, E)$, the **volume** of $S \subseteq V$ is $\text{Vol}(S) := \sum_{u \in S} \deg(u)$.

Definition 10. The **conductance** of $S \subseteq V$ is $\phi(S) := \frac{|\delta(S)|}{\min(\text{Vol}(S), \text{Vol}(V-S))}$.

Sparsest Cut Problem: Find a non-empty $S \subset V$ minimizing $\phi(S)$. Let $\text{OPT} := \min_S \phi(S)$.

It is noteworthy to mention that this is not always a min s - t cut, and since $|\delta(S)| \leq \min(\text{Vol}(S), \text{Vol}(V-S))$, we have $\phi(S) \leq 1$. Sparsest Cut is NP-hard.

2.2 H-Sparsest Cut

Definition 11. Given a complete edge-weighted graph $H = (V, E_H, w)$ for some $w : E \rightarrow \mathbb{R}_{\geq 0}$, the **H-conductance** of $S \subseteq V$ is $\phi_H(S) := \frac{|\delta_G(S)|}{w(\delta_H(S))}$, where $w(\delta_H(S)) = \sum_{e \in \delta_H(S)} w(e)$.

H-Sparsest Cut Problem: Find some non-empty $S \subset V$ minimizing $\phi_H(S)$.

Theorem 4. There exists a polynomial-time $O(\log n)$ -approximation for the H-Sparsest Cut problem. We will show this later on in the lecture.

Corollary 2. There exists a polynomial-time $O(\log n)$ -approximation for the Sparsest Cut problem.

Proof. $\forall u, v$, let $w_H(\{u, v\}) = \frac{\deg_G(u) \cdot \deg_G(v)}{m}$. We will show that $\phi(S) \leq \phi_H(S) \leq 2\phi(S) \forall S \subseteq V$. The corollary follows since the objective changed by at most a factor of 2.

Let $V_S = \text{Vol}(S)$, $\bar{V}_S = \text{Vol}(V - S)$. Then,

$$w(\delta_H(S)) = \sum_{u \in S, v \notin S} \frac{\deg_G(u) \cdot \deg_G(v)}{m} \quad (1)$$

$$= \frac{1}{m} \sum_{u \in S} \deg_G(u) \sum_{v \notin S} \deg_G(v) \quad (2)$$

$$= \frac{1}{m} \cdot V_S \cdot \bar{V}_S \quad (3)$$

Furthermore, since $\frac{m}{2} \leq \max(V_S, \bar{V}_S)$, we have

$$\frac{m}{2} \cdot \min(V_S, \bar{V}_S) \leq \min(V_S, \bar{V}_S) \cdot \max(V_S, \bar{V}_S) = V_S \cdot \bar{V}_S \leq m \cdot \min(V_S, \bar{V}_S),$$

so $w(\delta_H(S)) \in [\frac{1}{2} \min(V_S, \bar{V}_S), \min(V_S, \bar{V}_S)]$.

Therefore, $\forall S \subseteq V$, $\phi(S) \leq \phi_H(S) \leq 2 \cdot \phi(S)$. □

2.3 H-Sparsest Cut via Cut Metrics

There is a variable $d(u, v)$ for each $u, v \in V$. Let $d(e) := d(u, v)$ for $e = \{u, v\}$. Then, we have the following problem:

$$\min_d \frac{\sum_{e \in E} d(e)}{\sum_{u, v} d(u, v) \cdot w(\{u, v\})} \text{ s.t. } d \text{ is a cut metric on } V \quad (4)$$

However, this is not a Linear Program as the objective is a ratio and therefore not linear. Furthermore, the use of a cut metric gives integer constraints. The second can be relaxed by instead requiring d to be a “metric”, and adding the associated constraints for the metric space axioms (the Leighton-Rao Relaxation):

$$\begin{aligned}
\min_d \frac{\sum_{e \in E} d(e)}{\sum_{u,v} d(u,v) \cdot w(\{u,v\})} \text{ s.t. } & d(x,x) = 0 \forall x, \\
& d(x,y) = d(y,x) \forall x,y, \\
& d(x,y) \leq d(x,z) + d(z,y) \forall x,y,z
\end{aligned} \tag{5}$$

To finally make this into a linear program, we can require $\sum_{u,v} d(u,v) \cdot w(\{u,v\}) = 1$ to then make our objective a dot product:

$$\begin{aligned}
\min_d \sum_{e \in E} d(e) \text{ s.t. } & d(x,x) = 0 \forall x, \\
& d(x,y) = d(y,x) \forall x,y, \\
& d(x,y) \leq d(x,z) + d(z,y) \forall x,y,z, \\
& \sum_{u,v} d(u,v) \cdot w(\{u,v\}) = 1
\end{aligned} \tag{6}$$

Define OPT_{LR} to be the optimal value of this Linear Program.

Claim 6. $\text{OPT}_{LR} \leq \text{OPT}$

Proof. Let $\text{OPT}_{(i)}$ be the optimal value of objective (i) given above, and let $O_{(i)}$ be the objective of (i). We know that

$$\text{OPT}_{(5)} \leq \text{OPT}_{(4)} = \text{OPT}$$

as objective (5) minimizes over a larger set than objective (4). Thus, we want to show that

$$\text{OPT}_{LR} = \text{OPT}_{(6)} \leq \text{OPT}_{(5)}.$$

Thus, we want to show that $\text{OPT}_{(6)} \leq \text{OPT}_{(5)}$.

Suppose d is optimal for (5) and let $D = \sum_{u,v} d(u,v) \cdot w(\{u,v\})$. Then, $\text{OPT}_{(5)} = O_{(5)} = \frac{O_{(6)}}{D}$, but $\frac{1}{D} \cdot d$ is feasible for (6) so $\text{OPT}_{(6)} \leq O_{(6)} \left(\frac{1}{D} \cdot d\right) = \frac{O_{(6)}(d)}{D} = \text{OPT}_{(5)}$ \square

Now we can prove the original theorem. The idea is to use the optimal solution to (6), use Bourgain's Embedding to obtain an ℓ_1 metric (losing a factor of $O(\log n)$), to then use the ℓ_1 metric to find the cut that minimizes $\phi_H(S)$:

Algorithm 2 Approximate H-Sparsity Cut Algorithm

Let d be an optimal solution to (6)

Let (\tilde{V}, d_1) be what Bourgain embeds (V, d) into with $O(\log n)$ distortion

$d_1 \in \text{Cone}(\text{Cut}(\tilde{V}))$ so $d_1 = \sum_{S \in \mathcal{S}} \alpha_S \cdot d_S$ s.t. $\alpha_S \geq 0$ and d_S is a cut metric with cut S

Let S^* be the $S \in \mathcal{S}$ minimizing $\phi_H(S)$

return S^*

Both Bourgain's embedding and the construction of a polynomial-size cut decomposition of the resulting ℓ_1 metric can be performed in polynomial time. Since we only need to evaluate ϕ_H on these cuts, the resulting algorithm runs in polynomial time.

Let us show the approximation guarantee. For $S \in \mathcal{S}$, let $N_S := \sum_e d_S(e)$ and $D_S := \sum_{u,v} d_S(u,v) \cdot w(\{u,v\})$, so

$$S^* = \operatorname{argmin}_{S \in \mathcal{S}} \frac{N_S}{D_S} = \operatorname{argmin}_{S \in \mathcal{S}} \frac{\alpha_S \cdot N_S}{\alpha_S \cdot D_S}$$

Using that $\min_i \frac{a_i}{b_i} \leq \frac{\sum_i a_i}{\sum_i b_i}$ for non-negative a_i, b_i in the first step, we obtain:

$$\begin{aligned} \phi_H(S^*) &= \frac{N_{S^*}}{D_{S^*}} = \frac{\alpha_{S^*} N_{S^*}}{\alpha_{S^*} D_{S^*}} \\ &\leq \frac{\sum_S \alpha_S N_S}{\sum_S \alpha_S D_S} = \frac{\sum_e \sum_S \alpha_S \cdot d_S(e)}{\sum_{u,v} w(u,v) \sum_S \alpha_S \cdot d_S(u,v)} = \frac{\sum_e d_1(e)}{\sum_{u,v} w(u,v) \cdot d_1(u,v)} \end{aligned}$$

Thus, since Bourgain's Embedding has $O(\log n)$ distortion, we obtain:

$$\phi_H(S^*) \leq \frac{\sum_e d_1(e)}{\sum_{u,v} w(u,v) \cdot d_1(u,v)} \leq O(\log n) \cdot \frac{\sum_e d(e)}{\sum_{u,v} d(u,v) \cdot w(u,v)} = O(\log n) \cdot \operatorname{OPT}_{LR}$$

Thus, by the claim proven above, we have that

$$\phi_H(S^*) \leq O(\log n) \cdot \operatorname{OPT}_{LR} \leq O(\log n) \cdot \operatorname{OPT}$$

2.4 Expanders

Definition 12. $G = (V, E)$ is a ϕ -edge-expander, or ϕ -expander for short, if $\phi(S) \geq \phi$ for $\phi \in \mathbb{R} \forall S \subset V$ s.t. $S \neq \emptyset$.

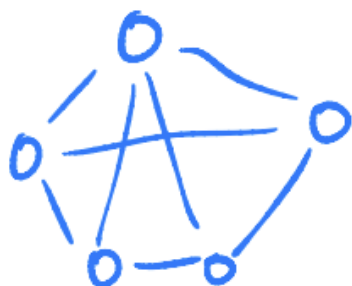


Figure 13: K_n (the complete graph on n vertices) is a $\Omega(1)$ -expander.

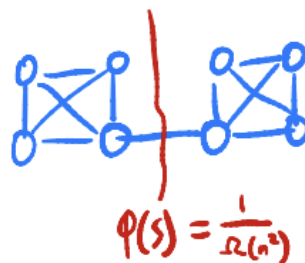


Figure 14: Above is a $O\left(\frac{1}{n^2}\right)$ -expander.

These are among the most well-studied graph classes in Theoretical Computer Science. Why? They're useful for proving impossibility results.

Fact 3. $\forall n_0$, there exists an $n \geq n_0$ -node 3-regular $\Omega(1)$ -expander, where 3-regular means that $\deg(v) = 3 \forall v$ (example given in 15). Therefore, surprisingly, there exist sparse graphs that are as connected as K_n .



Figure 15: An example 3-regular graph.

Expanders are used to show:

- LDD separation quality $\Omega\left(\frac{\log n}{\Delta}\right)$
- Embeddings into ℓ_1 or ℓ_2 require $\Omega(\log n)$ distortion (tightness of Bourgain)
- “Probabilistic tree embeddings” require $\Omega(\log n)$ distortion
- Many others...

They’re also useful for algorithms.

A ϕ -expander decomposition of $G = (V, E)$ is $F \subseteq E$ s.t. the connected components of $(V, E - F)$ are ϕ -expanders.

Fact 4. For all m -edge graphs, $\forall \phi \in (0, 1)$, there exists a ϕ -expander decomposition F s.t. $|F| \leq O(\phi \cdot \log n \cdot m)$.

Fact 5. The polynomial time α -approximate sparsest cut gives a polynomial time ϕ -expander decomposition F s.t. $|F| \leq O(\alpha \cdot \phi \cdot \log n \cdot m)$.