Today

1) Moving Around in LPs
2) Basic Feasible Solutions (BFSs)
3) Feasibility in exp. time via BFSs
4) Optimality of a BFS

Recall

LP Feasibility

Decide if $\exists x$ s.t. $Ax \le b$

$\rightarrow$

LP Search

Find x s.t. $Ax \le b$

$\rightarrow$

LP Optimization

Find x maxing $\langle c, x \rangle$ s.t. $Ax \le b$

$$\underbrace{T(N)}_{\le N^{2^N}}$$

improve to $\approx n^n$ then $\mathrm{Poly}(N)$

$\mathrm{Poly}(N) \cdot T(O(N))$

$\mathrm{Poly}(N) \cdot T(O(N))$

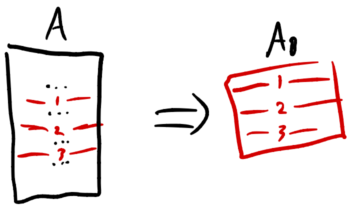
Tight Constraints

$x \in K$ is <u>tight</u> for constraint $\langle a_{i,j}, x\rangle \le b$ iff $\langle a_{i,j}, x\rangle = b$

$\text{Tight}_A(x) := \{a_i \in \text{rows}(A) : \langle a_{i,j}, x\rangle = b_i\}$

For $B \subseteq \text{rows}(A)$ let A_B

be matrix w/ rows B



$\text{rows}(A_B) = B$

Will consider $A_{\text{Tight}_A(x)} \hookrightarrow A_T$ for short

Will show following

Lemma: Given $x \in K$, $w \in \text{Ker}(A_T)$ $\exists \delta > 0$ s.t. $\underbrace{x \pm \delta \cdot w}_{\{x+\delta w,\, x-\delta w\}} \subseteq K$

and if $\langle w, a_i\rangle \neq 0$ for some $a_i \in \text{rows}(A)$ then $\text{Tight}(x) \subsetneq \text{Tight}(y)$

for some $y \in \underbrace{x \pm \delta \cdot w}_{\{x+\delta w,\, x-\delta w\}}$

Claim 1: Given $x \in K$, $w \in \mathbb{R}^n$ s.t. $\langle w, a_i \rangle \leq 0 \quad \forall a_i \in \text{rows}(A)$
have $x + \varepsilon \cdot w \in K \quad \forall \ \varepsilon \geq 0$

For $a_i \in \text{rows}(A)$, have $\langle a_i, x + \varepsilon w \rangle = \langle a_i, x \rangle + \varepsilon \cdot \langle a_i, w \rangle$

$$\leq \langle a_i, x \rangle$$

$$\leq b_i$$

Claim 2: Given $x \in K$, $w \in \text{Ker}(A_T)$ then $\text{Tight}(x) \leq \text{Tight}(x + \varepsilon \cdot w) \quad \forall \varepsilon \geq 0$

$w \in \text{Ker}(A_T) \rightarrow A_T w = 0 \rightarrow \langle a_i, w \rangle = 0 \quad \forall a_i \in T$

so $\forall \ a_i \in \text{Tight}(x)$ have $\langle a_i, x + \varepsilon w \rangle = \langle a_i, x \rangle + \varepsilon \langle a_i, w \rangle = \langle a_i, x \rangle = b_i$

so $a_i \in \text{Tight}(x + \varepsilon w)$

Lem 1: Given $x \in K$, $w \in \text{Ker}(A_T)$ w/ $\langle w, a_i \rangle \leq 0 \quad \forall a_i \in \text{rows}(A)$

have $x + \varepsilon w \in K$ and $\text{Tight}(x) \subseteq \text{Tight}(x + \varepsilon w) \quad \forall \ \varepsilon \geq 0$

Immediate from Claim 1+2

Lem 2 : Given $x \in K$, $w \in \mathrm{Ker}(A_T)$ w/ $\langle w, a_i \rangle > 0$ for some $a_i \in \mathrm{rows}(A)$

have $\exists \epsilon > 0$ s.t. $x + \epsilon' w \in K \quad \forall \ \epsilon' \in [0, \epsilon]$

and $\mathrm{Tight}(x) \subsetneq \mathrm{Tight}(x + \epsilon w)$

Let $I_w := \{ i : \langle a_i, w \rangle > 0 \}$, let $\epsilon_i := \dfrac{b_i - \langle x, a_i \rangle}{\langle w, a_i \rangle}$ for $i \in I_w$

Let $\epsilon = \min\limits_{i \in I_w} \epsilon_i$ and $y = x + \epsilon' w$ for $\epsilon' \leq \epsilon$

$y \in K$ b/c Consider an $a_i \in \mathrm{rows}(A)$

If $a_i \in T$ then $\langle y, a_i \rangle = b_i$ by Claim 2

If $a_i \notin T$

If $\langle a_i, w \rangle \leq 0$

$$\begin{aligned} \langle y, a_i \rangle &= \langle x, a_i \rangle + \epsilon' \langle w, a_i \rangle \\ &\leq \langle x, a_i \rangle \\ &= b_i \end{aligned}$$

If $\langle a_i, w \rangle > 0$

$$\begin{aligned} \langle y, a_i \rangle &= \langle x, a_i \rangle + \epsilon' \langle w, a_i \rangle \\ &\leq \langle x, a_i \rangle + \epsilon \cdot \langle w, a_i \rangle \quad (1) \\ &\leq \langle x, a_i \rangle + \epsilon_i \cdot \langle w, a_i \rangle \quad (2) \\ &= b_i \end{aligned}$$

For $i \in I_w$ s.t. $\epsilon_i = \epsilon$ (of which ≥ 1) and $\epsilon' = \epsilon$,

(1), (2) w/ equality so $\mathrm{Tight}(x) \subsetneq \mathrm{Tight}(x + \epsilon w)$

Lemma: Given $x \in K$, $w \in \mathrm{Ker}(A_T)$ $\exists \delta > 0$ s.t. $\underbrace{x \pm \delta \cdot w}_{\{x+\delta w,\, x-\delta w\}} \subseteq K$

and if $\langle w, a_i \rangle \neq 0$ for some $a_i \in \mathrm{rows}(A)$ then $\mathrm{Tight}(x) \subsetneq \mathrm{Tight}(y)$ for some $y \in \underbrace{x \pm \delta \cdot w}_{\{x+\delta w,\, x-\delta w\}}$

Say w type 1 if $w \in \mathrm{Ker}(A_T)$ and $\langle w, a_i \rangle \leq 0 \ \forall \ a_i \in \mathrm{rows}(A)$

type 2 if $w \in \mathrm{Ker}(A_T)$ and $\langle w, a_i \rangle > 0$ for some $a_i \in \mathrm{rows}(A)$

$w \in \mathrm{Ker}(A) \rightarrow -w \in \mathrm{Ker}(A)$ so each of $w, -w$ of type 1 or 2

Suppose both type 1

Lem 1 gives $x \pm \varepsilon w \subseteq K \ \forall \ \varepsilon \geq 0$

If $\langle w, a_i \rangle \neq 0$ for some $a_i \in \mathrm{rows}(A)$ then

at least 1 of $w, -w$ of type 2, WLOG w

Suppose w type 2, $-w$ type 1

Lem 2 gives ε s.t. $x + \varepsilon w \in K$ and $\mathrm{Tight}(x+\varepsilon w) \supsetneq \mathrm{Tight}(x)$

Lem 1 gives $x - \varepsilon w \in K$

Suppose both type 2

Lem 2 gives ε_1 s.t. $x + \varepsilon_1' w \in K \ \forall \varepsilon_1' \in [0, \varepsilon_1]$ and $\mathrm{Tight}(x+\varepsilon_1 w) \supsetneq \mathrm{Tight}(x)$

Lem 2 gives ε_2 s.t. $x - \varepsilon_2' w \in K \ \forall \varepsilon_2' \in [0, \varepsilon_2]$ and $\mathrm{Tight}(x-\varepsilon_2 w) \supsetneq \mathrm{Tight}(x)$

Let $\delta = \min(\varepsilon_1, \varepsilon_2)$

Basic Feasible Solutions

x is a basic feasible solution (BFS) of K iff $x \in K$ and $\text{rank}(\text{Tight}(x)) = n$

Trivial if 0 is feasible

↳ **Claim:** If $K = \left\{ \begin{matrix} Ax \le b \\ x \ge 0 \end{matrix} \right\} \ne \emptyset$ then K has a BFS

↳ Special Case: equational form LPs

If $A' = \begin{pmatrix} A \\ -I \end{pmatrix} = \begin{pmatrix} A \\ -e_1 \\ -e_2 \\ \vdots \\ -e_n \end{pmatrix}$, $b' = \begin{pmatrix} b \\ 0 \end{pmatrix}$ then $K = \{x : A'x \le b'\}$

Let $x \in K$ be any solution maximizing $|\text{Tight}_{A'}(x)| =: |T'|$

AFSOC x not a BFS so $\text{rank}(T') < n$ so $\text{rank}(A'_{T'}) < n$ LA FACT

$\exists w \ne 0$ s.t. $w \in \text{Ker}(A'_{T'})$ b/c $\text{rank}(A'_{T'}) < n$

$\exists a_i \in \text{rows}(A')$ s.t. $\langle a_i, w \rangle \ne 0$ b/c if $w_j \ne 0$ then $\langle -e_j, w_j \rangle \ne 0$

So by lemma $x \pm \varepsilon w \in K$ w/ $\text{Tight}(x) \subsetneq \text{Tight}(y)$ for $y \in x \pm \varepsilon w$

↳ Contradicts choice of x so x a BFS

Lemma: x is a BFS of $\{x : Ax \le b\}$

iff

$x \in K$ and $\exists$ basis $B \subseteq \text{rows}(A)$ of $\mathbb{R}^n$ w/ $A_B x = b_B$

$\rightarrow)$ $x \in K$ by definition
$\text{rank}(\text{Tight}_A(x)) = n$ so $\exists$ basis $B \subseteq \text{Tight}_A(x) \subseteq \text{rows}(A)$ of $\mathbb{R}^n$; also $A_B x = b_B$ b/c $B \subseteq \text{Tight}_A(x)$

$\leftarrow)$ $x \in K$ and since $A_B x = b_B$, have basis $B \subseteq \text{Tight}_A(x)$ so $\text{rank}(\text{Tight}_A(x)) = n$

Fact: If $B \subseteq \text{rows}(A)$ is a basis of $\mathbb{R}^n$ then $A_B x = b_B$ has 0 or 1 solution

$\text{rank}(A_B) = n$ so either 0 or 1 solution (Problem 7 on LA review)

Theorem: Can output all BFSs in $m^n \cdot \text{Poly}(N)$ time

$S = \emptyset$
$\forall\ B \in \binom{\text{rows}(A)}{n}$:
  If B is a basis and $\exists x$ s.t. $A_B x = b_B$ and $x \in K$
    $S \leftarrow S + x$
Return S

Correctness: x a BFS $\rightarrow \exists$ basis $B \subseteq \text{rows}(A)$ s.t. $A_B x = b_B$ and $x \in K \xrightarrow{\text{Fact}} x$ added to S
$x \in S \rightarrow \exists$ basis $B \subseteq \text{rows}(A)$ s.t. $A_B x = b_B$ and $x \in K \rightarrow x$ a BFS

Runtime: Check if B a basis in $\text{Poly}(N)$ time (Problem 8 on LA review $\rightarrow$ can check independence)
Find x s.t. $A_B x = b_B$ in $\text{Poly}(N)$ time (Fact 3.3 on LA review)
$\binom{m}{n} \le \left(\frac{me}{n}\right)^n \le m^n$ iterations

Theorem: Can solve LP Feasibility in $\text{Poly}(N) \cdot m^{2n}$ time

Put LP into form $\left\{x : \begin{array}{c} Ax \le b \\ x \ge 0 \end{array}\right\}$ → Note putting into above form doubles n

Let S be all BFSs of ↑

Return $K \ne \emptyset$ iff $S \ne \emptyset$

Correct by claim

Runtime from theorem

Claim: If $K = \left\{ \begin{matrix} Ax \le b \\ x \ge 0 \end{matrix} \right\} \neq \emptyset$ and $\max_{x \in K} \langle c, x \rangle \neq \infty$

then K has a BFS y w/ $\langle c, y \rangle = \max_{x \in K} \langle c, x \rangle$

If $A' = \begin{pmatrix} A \\ -I \end{pmatrix} = \begin{pmatrix} A \\ -\!-e_1 - \\ -\!-e_2 - \\ \vdots \\ -\!-e_n - \end{pmatrix}$, $b' = \begin{pmatrix} b \\ 0 \end{pmatrix}$ then $K = \{x : A'x \le b'\}$

Let $T' := \text{Tight}_{A'}(x)$

Let $x \in K$ be any optimal solution maximizing $|\text{Tight}_{A'}(x)| =: |T'|$

AFSOC x not a BFS so $\text{rank}(T') < n$ so $\text{rank}(A'_{T'}) < n$

$\exists w \neq 0$ s.t. $w \in \text{Ker}(A'_{T'})$ b/c $\text{rank}(A'_{T'}) < n$

$\exists a_i \in \text{rows}(A')$ s.t. $\langle a_i, w \rangle \neq 0$ b/c if $w_j \neq 0$ then $\langle -e_j, w_j \rangle \neq 0$

So by lemma $x \pm \varepsilon w \in K$ w/ $\text{Tight}(x) \subsetneq \text{Tight}(y)$ for $y \in x \pm \varepsilon w$

$\langle c, w \rangle = 0$ b/c $\langle c, w \rangle \neq 0 \to \langle c, w \rangle > 0$ or $\langle c, -w \rangle > 0$

$\to \langle x + \varepsilon w, c \rangle > OPT$ or $\langle x - \varepsilon w, c \rangle > OPT$

Contradicting OPT definition

But then $y \in K$, $\langle c, y \rangle = OPT$ and $\text{Tight}(x) \subsetneq \text{Tight}(y)$

Contradicting choice of x

$\hookrightarrow$ Note: outputting best BFS also then gives an optimization alg.