

Today

- 1) Max bipartite matching
- 2) Integer Programming + LP Integrality
- 3) The LP Framework
- 4) Solving (1) w/ (3)

Recall

Claim: If $K = \{x: Ax \leq b, x \geq 0\}$ then K has a BFS

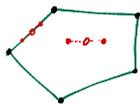
Further, given c w/ $OPT := \max_{y \in K} \langle c, y \rangle \neq \infty$

then \exists a BFS z s.t. $\langle c, z \rangle = OPT$

and z can be computed in $\text{poly}(N)$ time

Fact: $x \in K = \{Ax \leq b\}$ is a BFS iff it is a vertex iff it is an extreme point

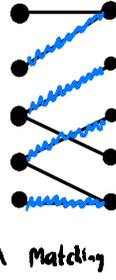
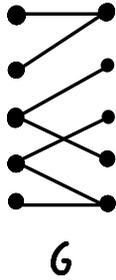
Given $K \subseteq \mathbb{R}^n$, x is an extreme point of K if $x \in [y, z]$ for $y, z \in K \rightarrow y = z$



Max Bipartite Matching

Given bipartite graph $G=(V,E)$, find a matching $M \subseteq E$ maximizing $|M|$

↳ No 2 distinct edges share an endpoint



An Equivalent Problem

Let $E = \{e_1, e_2, \dots, e_n\}$

For $x \in \mathbb{R}^n$, notate x_i as x_{e_i}

Call this
Problem
 IP_M

Find $x \in \mathbb{R}^n$ maximizing $\sum_{e \in E} x_e$ s.t.

$$x(\delta(u)) \leq 1 \quad \forall u \in V$$

$$x_e \in \{0, 1\} \quad \forall e \in E$$

1-1 Correspondence between these x and matchings

$$x \rightarrow \text{Matching } \{e : x_e = 1\}$$

$$M \rightarrow x \text{ where } x_e = \begin{cases} 1 & \text{if } e \in M \\ 0 & \text{o/w} \end{cases}$$

Not a linear program b/c $x_e \in \{0, 1\}$ not a linear constraint

IP_M is an "integer" program

Integer Programming

Suppose given $K = \{x: Ax \leq b\}$ and $c \in \mathbb{R}^n$

\uparrow
n x n

let $OPT_{LP}(c) := \max_{x \in K} \langle c, x \rangle$ and $OPT_{IP}(c) := \max_{x \in K \cap \mathbb{Z}^n} \langle c, x \rangle$

(or report $K \cap \mathbb{Z}^n = \emptyset$ or $OPT_{IP}(c) = -\infty$)

Given c , find $x \in K \cap \mathbb{Z}^n$ s.t. $\langle c, x \rangle = OPT_{IP}(c)$ is called an integer program

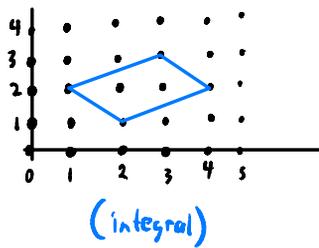
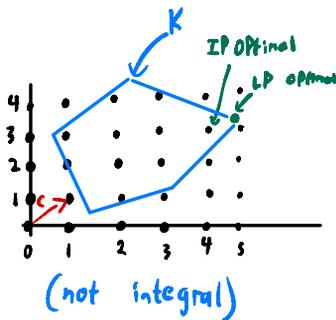
Bad news: integer programming is NP-hard

(or report...)

Given c , find $x \in K$ s.t. $\langle c, x \rangle = OPT_{LP}(c)$ is called the LP relaxation of

Good news: LPEP

Question then is how well the LP relaxation approx the IP



K is integral iff all BFS of K are integral ($x \in \mathbb{R}^n$ is integral if $x \in \mathbb{Z}^n$)

Given a set of Polyhedra Π and costs c the integrality gap is

$$\max_{\substack{K \in \Pi \\ c \in C}} \frac{OPT_{LP}(c)}{OPT_{IP}(c)}$$

E.g. $c = 1$ and K is all Polyhedra corresponding to instances of Max matching

$$OPT_{LP}(c) \geq OPT_{IP}(c) \quad \forall c \text{ so } IG \geq 1$$

Note: if all $K \in \Pi$ integral then $IG = 1$

Skip
for
time

The LP Framework

- 1) Express Problem as an IP
- 2) Get a BFS solution x to LP relaxation of (must assume 1 exists)
- 3) Use x to solve Problem (sometimes only approximately)

↳ Simplest (3) is when K is integral b/c x is a solution to IP and so to Problem

Back to Matching

$$\text{Let } K_M := \left\{ x \in \mathbb{R}^m : \begin{array}{l} x(\delta(u)) \leq 1 \quad \forall u \in V \\ x_e \in [0, 1] \quad \forall e \in E \end{array} \right\}$$

So $IP_M \Leftrightarrow$ find $x \in K_M \cap \mathbb{Z}^m$ s.t. $\langle 1, x \rangle = IP_{opt}(1)$

w/ LP relaxation find $x \in K_M$ s.t. $\langle 1, x \rangle = LP_{opt}(1)$

Claim: K_M is integral

↓ gives

Algorithm for Matching

- (1) Consider IP_M
- (2) Let $y \in K_M$ be a BFS s.t. $\langle 1, y \rangle = \max_{x \in K} \langle 1, x \rangle$
- (3) Return matching corresponding to y

Always exists b/c
1) $LP_{opt} \neq \infty$
2) $x \geq 0$ in constraints

Proof that K_M is integral

Let y be a BFS of K_M

Say $e \in E$ is fractional if $y_e \in (0,1)$, let $F := \{e \in E : y_e \in (0,1)\}$, AFSOC $|F| \geq 1$

Say vertex $u \in V$ is tight iff $y(\delta(u)) = 1$

$\forall \{u,v\} \in E$ fractional, either u or v or both are tight

AFSOC $e = \{u,v\}$ fractional but u and v not tight

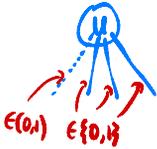


Let $\epsilon = \min(1 - y(\delta(u)), 1 - y(\delta(v)), y_e) \rightarrow$ Note $\epsilon > 0$

Let $x_e := (0, 0, \dots, \underset{\substack{\uparrow \\ \text{position of } e}}{1}, \dots, 0) \in \mathbb{R}^E$ and let $y^+ := y + \epsilon \cdot x_e$ and $y^- := y - \epsilon \cdot x_e$

Then $y^-, y^+ \in K_M$, $y^- \neq y^+$ but $\frac{1}{2} \cdot y^+ + \frac{1}{2} \cdot y^- = y$, contradicting y an extreme point
 $y^- \in K_M$ b/c $\epsilon < y_e$, $y^+ \in K_M$ b/c $y^+(\delta(u)) = y(\delta(u)) + \epsilon = 1$, v symmetric

If $\{u,v\}$ is fractional and u is tight, then $|\delta(u) \cap F| \geq 2$



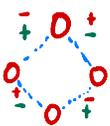
\rightarrow b/c $1 = y(\delta(u)) = \sum_{e \in \delta(u)} y_e$ and can't \sum_i exactly one $\epsilon \in (0,1)$ w/ other $\#s \in \{0,1\}$

to get 1

(LUR, F) either has a cycle or a path (u, \dots, v) where u and v not tight
 start at a fractional edge and follow edges out of tight endpoints



Suppose a cycle C



Must have even $\#$ edges
 b/c bipartite \rightarrow each vtx. in to 1 even, 1 odd edge
 Let y^+ add ϵ to even edges
 $-\epsilon$ to odd edges
 y^- add $-\epsilon$ to even edges
 ϵ to odd edges

For $\epsilon = \min(y_e, 1 - y_e) > 0$, some contradiction as before

Suppose a path P



Let y^+, y^- be as in cycle case
 but w/ $\epsilon = \min(y_e, 1 - y(\delta(u)), 1 - y(\delta(v)))$
 $e \in P$

Same contradiction as before ($y^+, y^- \in K_M$, $y^+ \neq y^-$ and $\frac{1}{2} \cdot y^+ + \frac{1}{2} \cdot y^- = y$)

Bonus: Can solve max weight bipartite matching in some way

Given bipartite graph $G=(L,R,E)$ $w: E \rightarrow \mathbb{R}$ find a matching $M \subseteq E$ maximizing $\sum_{e \in M} w(e)$

Some argument, just replace 1 w/ w

Other Common Integral LPs

- Flow (on hw) \rightarrow generalized by TU matrix + network matrix LPs
- Cut (on hw)
- MST \rightarrow generalized by matroid LPs
- General graph matching (on hw)
- Arborescence (directed MST) \rightarrow generalized by matroid intersection LPs