Today

1) LPs as shapes
2) Inner products + friends
3) Ellipsoid Algorithm

LPs as a "Shape"

$$K = \{x : Ax \leq b\} = \{x : \langle a_i, x \rangle \leq b_i \quad \forall\ a_i \in \text{rows}(A)\}$$

A bunch of points in $\mathbb{R}^n$
What kind of "shape"?

A set of the form $K = \{x : Ax \leq b\}$ is called a <u>Polyhedron</u>

If $\exists\ \Delta \in \mathbb{R}$ s.t. $d(x,y) \leq \Delta \ \ \forall x, y \in K$, then K is called a <u>Polytope</u>

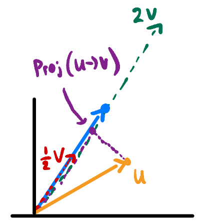
↳ K is "bounded"

Inner Products + Friends

Geometry of $\langle u, u \rangle$: $\sqrt{\langle u, u \rangle} := \sqrt{\sum_i (u_i)^2} = d(0, u) = \text{length of } u$

↳ Notated $\|u\|$

Geometry of $\langle u, v \rangle$: $\langle u, v \rangle := \sum_i u_i v_i = \ ?$



Let $c^* := \arg\min_c d(c \cdot v, u)$ and $\text{Proj}(u \to v) := c^* \cdot v$

Do some calculus and get $c^* = \dfrac{\langle u, v \rangle}{\langle v, v \rangle}$

so $\text{Proj}(u \to v) = \langle u, v \rangle \dfrac{v}{\langle v, v \rangle}$

$\langle u, v \rangle \longleftrightarrow$ how much u is in direction of v

<u>Some calculus:</u>

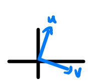
$\min_c d(c \cdot v, u) = \min_c \left(d(c \cdot v, u)\right)^2$ b/c $d \geq 0$ and squaring is monotone increasing

So let $f(c) = \left(d(c v, u)\right)^2 = \sum_j (c \cdot v_j - u_j)^2 = \sum_j c^2 v_j^2 - 2c \cdot v_j u_j + u_j^2$

$\dfrac{df}{dc} = 2 \sum_j c v_j^2 - 2 v_j u_j$

Set $\dfrac{df}{dc} = 0$, solve for c to get $c = \dfrac{\langle u, v \rangle}{\langle v, v \rangle}$

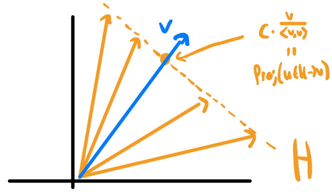
Then do 2nd derivative test

$\langle u, v \rangle = 0 \rightarrow$  $\Rightarrow u, v$ orthogonal

Given $v \in \mathbb{R}^n$, $c \in \mathbb{R}$ $H_{=v,c} := \{u : \langle u, v\rangle = c\}$ is called a **hyperplane**

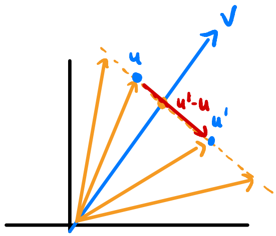
Equivalently, $H_{=v,c} = \{u : \text{Proj}(u \to v) = c \cdot \frac{v}{\langle v,v\rangle}\}$

↳ so $u, u' \in H_= \to \text{Proj}(u \to v) = \text{Proj}(u' \to v)$

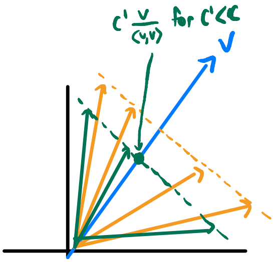


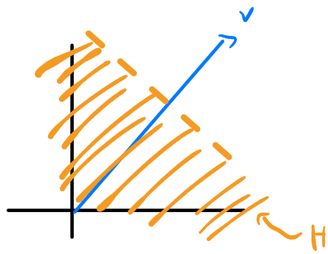
v is called the **normal vector** of $H_=$

B/c if $u, u' \in H_=$ then $\langle u' - u, v\rangle = \langle u', v\rangle - \langle u, v\rangle = c - c = 0$

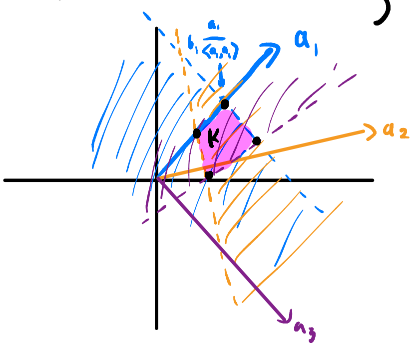


Given $v \in \mathbb{R}^n$, $c \in \mathbb{R}$ $H := \{u : \langle u, v\rangle \le c\}$ is called a **halfspace**





$K = \{x : \langle a_i, x\rangle \le b_i \;\; \forall\, a_i \in \text{rows}(A)\}$ is the intersection of m halfspaces



Notice if x is tight for

$\langle a_i, x\rangle \le b_i$

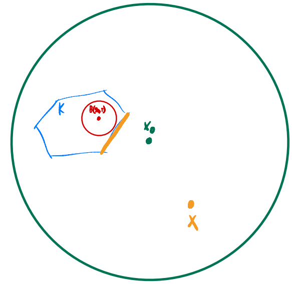
then x is in the hyperplane $\{x : \langle a_i, x\rangle = b_i\}$

Also notice if only 2 constraints here get polyhedra

Main Takeaway: Can solve LP feasibility in Polynomial time (even w/ exponentially-many constraints)

Assumptions

1) Given x_0, R s.t. $K \subseteq B(x_0, R)$
2) $\exists\, y_0, r \in \mathbb{R}$ s.t. if $K \neq \emptyset$ $B(y_0, r) \subseteq K$ w/ $\frac{R}{r} \le \exp(\text{Poly}(n))$
3) Have a "Strong Separation oracle" for K (in poly(n) time)



Given $y \in \mathbb{R}^n$, a strong separation oracle for $K = \{x : Ax \le b\}$ either correctly outputs "$y \in K$" or returns an $a_j \in \text{rows}(A)$ s.t. $\langle y, a_j \rangle > b_j$ → Note $\langle x, a_j \rangle \le b_j \;\; \forall x \in K$

Ellipsoid Algorithm

Let $E_0 := B(x_0, R)$

For $i = 1, 2, \dots, 10 \cdot n^2 \cdot \ln\left(\frac{R}{r}\right)$

  Let $c_{i-1} := \text{center}(E_{i-1})$

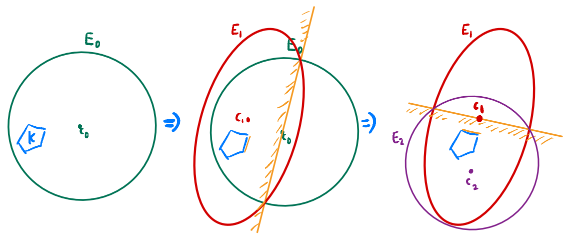
  If $c_{i-1} \in K$, return $K \neq \emptyset$

  O/w let $a_j \in \text{rows}(A)$ satisfy $\langle a_j, c_{i-1} \rangle > b_j$ → via separation oracle

  Let $T_i := E_{i-1} \cap \{x : \langle a_j, x \rangle \le \langle a_j, c_{i-1} \rangle\}$

  Let E_i be a carefully chosen "ellipsoid" containing T_{i-1} → will be min volume ellipsoid

Return $K = \emptyset$



Volume + Balls

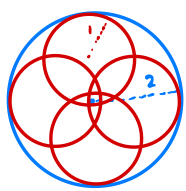
For $S \subseteq \mathbb{R}^n$, the volume of S is $\text{Vol}(S) := \int_{x \in S} 1\, dx$

The (closed) radius r ball centered at $x \in \mathbb{R}^n$ is

$$B(x, r) := \{y \in \mathbb{R}^n : d(x, y) \le r\}$$

Let $V_n := \text{Vol}(B(x, 1))$

Fact: $\text{Vol}(B(x, r)) = V_n \cdot r^n \quad \forall r \ge 0,\ x \in \mathbb{R}^n$



Key Lemma: Can Poly(n)-time compute E_i s.t. $T_i \subseteq E_i$ and $\text{vol}(E_i) \le \left(1 - \frac{1}{5n}\right) \cdot \text{vol}(E_{i-1})$

Thm: Given (1)-(3), ellipsoid solves LP Feasibility in Poly(n) time

Proof of Ellipsoid Correctness → A halving argument!

If $K=\emptyset$ then never return $K\neq\emptyset$

If $K\neq\emptyset$, know $\exists r, y_0$ as described, AFSOC returned $K=\emptyset$

Let $k := 10\cdot n^2\cdot \ln\left(\frac{R}{r}\right)$ be # of iterations

$\text{Vol}(E_k)\geq V_n\cdot r^n$ b/c key lemma → $K\subseteq E_i\ \forall i$ and $B(y_0,r)\subseteq K$

so $B(y_0,r)\subseteq E_k$ but $\text{Vol}(B(y_0,r))=V_n\cdot r^n$ so $\text{Vol}(E_k)\geq V_n\cdot r^n$

OTOH, by key lemma and $\text{Vol}(E_0)=V_n\cdot R^n$ have

$$\text{Vol}(E_k)\leq\left(1-\tfrac{1}{5n}\right)\cdot\text{Vol}(E_{k-1})\leq\left(1-\tfrac{1}{5n}\right)^2\cdot\text{Vol}(E_{k-2})\leq\ldots\leq\text{Vol}(E_0)\cdot\left(1-\tfrac{1}{5n}\right)^k$$

$$=V_n\cdot R^n\left(1-\tfrac{1}{5n}\right)^k$$

$$\leq V_n\cdot R^n\exp(-k/5n)\qquad (1-x\leq\exp(-x)\ \forall x)$$

$$=V_n\cdot R^n\cdot\exp\left(-2\cdot n\cdot\ln\left(\tfrac{R}{r}\right)\right)$$

$$=V_n\cdot R^n\cdot\left(\frac{r}{R}\right)^{2n}$$

$$=V_n\cdot\left(\frac{r}{R}\right)^n\cdot r^n$$

$$<V_n\cdot r^n \rightarrow \text{contradicts } \text{Vol}(E_k)>V_n\cdot r^n$$

Proof of Ellipsoid Runtime

Runtime is $T_{\text{separation}}\cdot O\left(n^2\cdot\ln\left(\frac{R}{r}\right)\right)=\text{Poly}(n)$ $\left(\text{by } \frac{R}{r}\leq\exp(\text{Poly}(n))\right)$