Today

- LP Feasibility Duality (Farkas lemma) ②
- LP Optimization Duality ③
- E.g. of max flow
- Hyperplane Separation Theorems ①
- ① → ②
- ② → ③
- Proof of ①

Recall

Fact: Given $B : \mathbb{R}^n \to \mathbb{R}$, $b \in \mathbb{R}$

If $x \in \mathbb{R}^n$ satisfies $B(x) \le b$

then $\forall\, \lambda \in \mathbb{R}_{\ge 0}$ x satisfies $\lambda \cdot B(x) \le \lambda \cdot b$

Fact: Given $B_1, B_2, \dots, B_m : \mathbb{R}^n \to \mathbb{R}$, $b_1, b_2, \dots, b_m \in \mathbb{R}$

If $x \in \mathbb{R}^n$ satisfies $B_i(x) \le b_i \quad \forall i \in [m]$

then x satisfies $\sum_i B_i(x) \le \sum_i b_i$

Feasibility Duality

Feasibility Goal: give certificate of non-feasibility

$$\exists x \text{ s.t. } \begin{aligned} \langle a_1, x\rangle &\le b_1 \\ \langle a_2, x\rangle &\le b_2 \\ &\cdots \\ \langle a_m, x\rangle &\le b_m \end{aligned}$$

$\big\downarrow$ only if $\forall \lambda \in \mathbb{R}^n_{\ge 0}$

$$\exists x \text{ s.t. } \begin{aligned} \lambda_1\langle a_1, x\rangle &\le \lambda_1 \cdot b_1 \\ \lambda_2\langle a_2, x\rangle &\le \lambda_2 \cdot b_2 \\ &\cdots \\ \lambda_m\langle a_m, x\rangle &\le \lambda_n \cdot b_m \end{aligned}$$

$\big\downarrow$ Only if $\forall \lambda \in \mathbb{R}^n_{\ge 0}$

$$\exists x \text{ s.t. } \Big\langle \sum_i \lambda_i a_i, x\Big\rangle \le \sum_i \lambda_i b_i$$

$$\exists x \text{ s.t. } Ax \le b$$

$\big\downarrow$ Only if $\forall \lambda \in \mathbb{R}^n_{\ge 0}$

$$\exists x \text{ s.t. } \langle A^T\lambda, x\rangle \le \langle \lambda, b\rangle$$

So, if $\exists \lambda$ s.t. $\langle A^T\lambda, x\rangle > \langle \lambda, b\rangle$ $\forall x$ then $\nexists x$ s.t. $Ax \le b$

Suppose $\exists \lambda \in \mathbb{R}^n_{\ge 0}$ s.t. $A^T\lambda = 0$ and $\langle \lambda, b\rangle < 0$

Then $\forall x \in \mathbb{R}^n$ have $\langle A^T\lambda, x\rangle = \langle 0, x\rangle = 0 > \langle \lambda, b\rangle$

I.e. $\nexists x$ s.t. $Ax \le b$

Lem: If $\exists\ \lambda \in \mathbb{R}^m_{\ge 0}$ s.t. $A^T\lambda = 0$ but $\langle \lambda, b\rangle < 0$ then $\nexists x$ s.t. $Ax \le b$

iff version

Farkas Lemma: $\exists\ \lambda \in \mathbb{R}^m_{\ge 0}$ s.t. $A^T\lambda = 0$ and $\langle \lambda, b\rangle < 0$ iff $\nexists x$ s.t. $Ax \le b$

Many variants

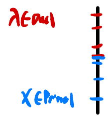
Optimization Duality

- Optimization Goal: give certificate of (tight) upper bound of optimal value

Consider LP optimization

$$\max \langle c, x\rangle \text{ s.t. } Ax \le b \qquad \min \langle b, \lambda\rangle \text{ s.t. } A^T\lambda = c^T,\ \lambda \ge 0$$

(Primal) (Dual)



Let $P := \max_{x : Ax \le b} \langle c, x\rangle$

Let $D := \max_{\substack{\lambda : A^T\lambda = c^T \\ \lambda \ge 0}} \langle b, \lambda\rangle$

b/c $Ax \le b \to \langle A^T\lambda, x\rangle \le \langle \lambda, b\rangle \ \forall \lambda \ge 0$ as on previous page

Suppose $\exists \lambda \ge 0$ s.t. $A^T\lambda = c$

Then $\forall x$ s.t. $Ax \le b$ have $\langle c, x\rangle = \langle A^T\lambda, x\rangle \le \langle \lambda, b\rangle$

I.e. $P \le \langle \lambda, b\rangle$ for this λ

How to choose λ? → find best λ w/ dual LP as above!

Weak Duality: Always have $P \le D$

Proof is above

Primal infeasible → Dual unbounded
Primal unbounded → Dual infeasible

Strong Duality: If Primal feasible + bounded then $P = D$

In dual, objective swaps w/ upper bound, constraints swap w/ variables

Primal:	Max	Objective $\langle x, \underline{c} \rangle$	Upper bound $Ax \leq \underline{b}$	Constraint $\langle a_{i,}, x \rangle \leq b_i$ $\forall i \in [m]$	Variable X_i for $i \in [n]$
Dual:	Min	Upper bound $A^T \lambda = \underline{c}^T$	Objective $\langle \lambda, \underline{b} \rangle$	Variable λ_i $\forall i \in [m]$	Constraint $A^T \lambda = c^T$

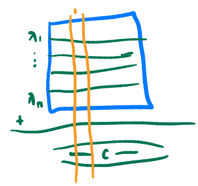
$A^T \lambda = c^T$

iff

$(A^T \lambda)_i = c_i^T \quad \forall i \in [n]$

iff

$\sum_{j=1}^{m} A_{ji} \cdot \lambda_j = c_i \quad \forall i \in [n]$



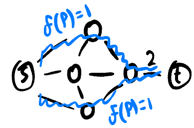
Primal + Dual Example

Maximum flow

Given graph $G=(V,E)$, capacities $u: E \to \mathbb{Z}_{\geq 0}$, $s,t \in V$

Let $\mathcal{P}(s,t)$ be all s-t paths

Want $f: \mathcal{P}(s,t) \to \mathbb{R}_{\geq 0}$ maximizing $\sum_P f(P)$ s.t. $\forall e \in E$ $\sum_{P: e \in P} f(P) \leq u(e)$



Max flow as an LP

Variable x_P $\forall P \in \mathcal{P}(s,t)$ so $x \in \mathbb{R}^k$ where $k := |\mathcal{P}(s,t)|$

$$\max \sum_P x_P \text{ s.t.}$$

$$\sum_{P: e \in P} x_P \leq u(e) \quad \forall e \in E$$

$$x_P \geq 0 \quad \forall P \in \mathcal{P}(s,t)$$

$$\Downarrow$$

Want in form $\langle c, x \rangle$ s.t. $Ax \leq b$ to apply duality

$$\max \langle \overset{c}{\mathbb{1}}, x \rangle \text{ s.t.}$$

$$\begin{array}{cc} & \begin{matrix} P_1 & P_2 & \cdots & P_j & \cdots & P_k \end{matrix} \\ \begin{matrix} \lambda_{e_1} & e_1 \sim \\ \vdots & \vdots \\ \lambda_{e_i} & e_i \sim \\ \vdots & \vdots \\ \lambda_{e_m} & e_m \sim \\ \lambda'_1 & P_1 \sim \\ \lambda'_2 & P_2 \sim \\ \vdots & \vdots \\ \lambda'_k & P_k \sim \end{matrix} & \underbrace{\begin{pmatrix} & & & & & \\ & & & \boxed{\mathbb{1}(e_i \in P_j)} & & \\ & & & & & \\ & & & & & \\ & & & & & \\ -1 & & & & & 0 \\ & -1 & & & & \\ 0 & & \ddots & & & \\ & & & & & -1 \end{pmatrix}}_{A} \end{array} \begin{pmatrix} x_{P_1} \\ \vdots \\ \\ \\ \\ \\ \\ \\ x_{P_k} \end{pmatrix} \leq \underbrace{\begin{pmatrix} u(e_1) \\ u(e_2) \\ \vdots \\ u(e_m) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{b}$$

Let $p :=$ max flow value (optimal to above LP)

Max Flow LP Dual

Min $\sum_e \lambda_e \cdot u(e)$ s.t.

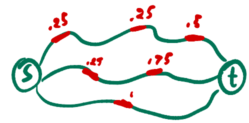
$$\sum_{e \in P_i} \lambda_e - \lambda'_i = 1 \quad \forall P_i \in \mathcal{P}(s,t) \quad \longleftrightarrow$$

$$\lambda, \lambda' \geq 0$$

Min $\sum_e \lambda_e \cdot u(e)$ s.t.

$$\sum_{e \in P_i} \lambda_e \geq 1 \quad \forall P_i \in \mathcal{P}(s,t)$$

$$\lambda \geq 0$$



cost is given by u

"Min cost assignment of values to edges so every s-t path gets ≥ 1"

Let D_F be optimal dual value so $P_F = D_F$

Can turn into a more interesting structural fact

Say $F \subseteq E$ <u>disconnects</u> s,t if s,t not connected in $(V, E \setminus F)$

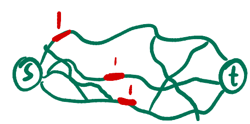
Call $u(F) := \sum_{e \in F} u(e)$ the cost of F

Correspondence: $\lambda \in \{0,1\}^{|E|}$ feasible for dual $\xrightarrow{F = \{e : \lambda_e = 1\}}$ $F \subseteq E$ that disconnects s,t

$\longleftarrow$ $\lambda_e = 1$ iff $e \in F$

<u>Fact:</u> If λ optimal then $\lambda \leq 1$

O/w can decrease λ to reduce cost



<u>Fact:</u> Above dual LP is integral

Will see on homework

<u>Fact:</u> $\exists$ $\lambda \in \{0,1\}^n$ s.t. λ feasible and optimal for flow dual

By above facts

<u>Fact:</u> $P_F = \min_{F \text{ disconnecting } s,t} u(F)$ $\longleftarrow$ Equivalent to min s-t cut (will see on homework) so max flow = min cut

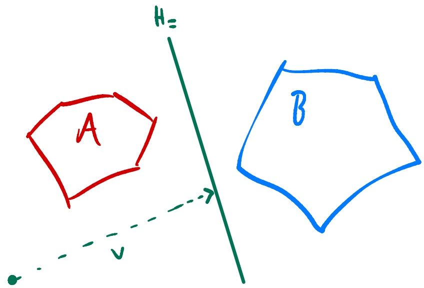
$P_F = D_F = \min_{F \text{ dis. } s,t} u(F)$

Hyperplane Separation

Given $A, B \subseteq \mathbb{R}^n$, $H_= = \{u : \langle u, v\rangle = c\}$ **strictly separates** A and B if

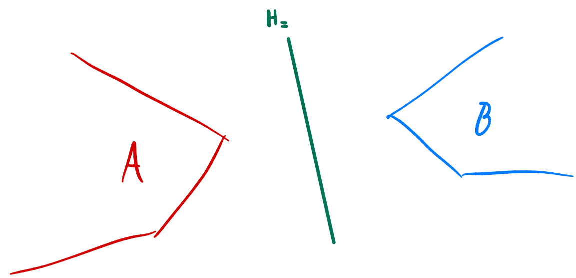
$$\langle v, a\rangle < c < \langle v, b\rangle \quad \forall a \in A, b \in B$$

Note: suffices to find v s.t. $\langle v, a\rangle < \langle v, b\rangle \quad \forall a \in A, b \in B$



Polyhedral Separation

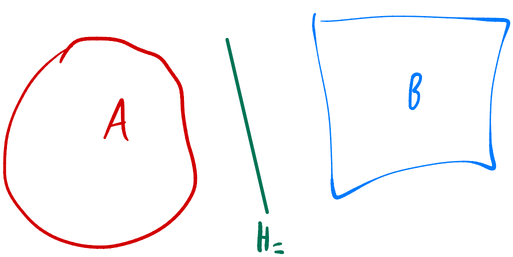
Given disjoint non-empty polyhedra A, B, $\exists$ a hyperplane that strictly separates A, B



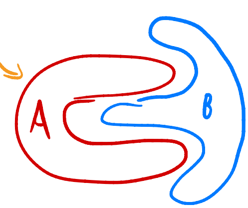
Convex Separation

Note: not immediate from above b/c polyhedra can be unbounded

Given disjoint, non-empty $A, B \subseteq \mathbb{R}^n$ convex (A closed, B compact) $\exists$ a hyperplane $H_=$ that strictly separates A, B



Not true if not convex



Got to here in class

Proof of Convex Separation

Given A (closed), B (compact) both convex

Let $a_0 \in A$ and $b_0 \in B$ be 2 points minimizing $d(a,b)$

Euc. distance

Proof $a_0, b_0 \exists$

Let $\Delta := \inf_{a \in A, b \in B} d(a,b)$

For $b \in B$, let $f(b) := \inf_{a \in A} d(a,b)$

Since f continuous, B compact, f attains min @ some a_0

Let $\tilde{B}$ be $B \cap \mathrm{Ball}(a_0, 2\Delta) \to \tilde{B}$ compact $+ \neq \emptyset$

Let $g(b) := d(a_0, b) \to g$ continuous so min attained on $\tilde{B}$ @ b_0

Claim that $H_= := \left\{x : \langle b_0 - a_0, x \rangle = \frac{\|b_0\|^2 - \|a_0\|^2}{2}\right\}$ Strictly separates A, B

Since $A \cap B = \emptyset$, know $b_0 - a_0 \neq 0$ so $\|b_0 - a_0\| > 0$ so $\langle b_0 - a_0, b_0 - a_0 \rangle > 0$

so $\langle a_0, b_0 - a_0 \rangle < \langle b_0, b_0 - a_0 \rangle$

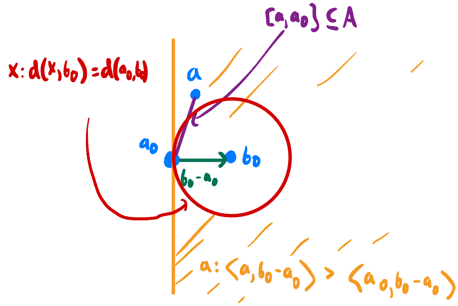
Thus, suffices to show $\forall a \in A$ have $\langle a, b_0 - a_0 \rangle \leq \langle a_0, b_0 - a_0 \rangle$

and $\forall b \in B$ have $\langle b_0, b_0 - a_0 \rangle \leq \langle b, b_0 - a_0 \rangle$

Both cases symmetric so wts $\forall a \in A$ have $\langle a, b_0 - a_0 \rangle \leq \langle a_0, b_0 - a_0 \rangle$

AFSOC $\exists a \in A$ s.t. $\langle a, b_0 - a_0 \rangle > \langle a_0, b_0 - a_0 \rangle$

Rest of proof by pictures:



By convexity $[a, a_0] \subseteq A$ but then $\exists x \in [a, a_0] \subseteq A$ s.t. $d(x, b_0) < d(a_0, b_0)$ $\rightarrow\leftarrow$ a_0 choice

Rest of proof by calculation:

Know $[a, a_0] \subseteq A$ by convexity; let $a_\varepsilon := (1-\varepsilon)a_0 + \varepsilon \cdot a$ $\quad (a_\varepsilon \in A \ \forall \varepsilon \in [0,1])$

So $d(a_\varepsilon, b_0) = \|b_0 - (1-\varepsilon)a_0 - \varepsilon a\| = \|(1-\varepsilon)(b_0 - a_0) + \varepsilon(b_0 - a)\|$

$$= \sqrt{(1-\varepsilon)\|b_0 - a_0\|^2 + \varepsilon\|b_0 - a\|^2 + (1-\varepsilon)(\varepsilon) \cdot \langle b_0 - a_0, b_0 - a \rangle}$$

$$< \sqrt{(1-\varepsilon)\|b_0 - a_0\|^2 + \varepsilon\|b_0 - a\|^2 + (1-\varepsilon)(\varepsilon) \cdot \langle b_0 - a_0, b_0 - a_0 \rangle}$$

$$= \sqrt{(1-\varepsilon)\|b_0 - a_0\|^2 + \varepsilon\|b_0 - a\|^2 + (\varepsilon - \varepsilon^2)\|b_0 - a_0\|^2}$$

$$= \sqrt{\|b_0 - a_0\|^2 + \varepsilon\|b_0 - a\|^2 - \varepsilon^2\|b_0 - a_0\|^2}$$

$$\leq \|b_0 - a_0\| \text{ for } \varepsilon \text{ sufficiently small}$$

So $\exists \varepsilon$ s.t. $a_\varepsilon \in A$ and $d(a_\varepsilon, b_0) < d(a_0, b_0)$ $\rightarrow\leftarrow$ choice of a_0

Proof of Polyhedral Separation w/ Convex Separation

Note if $A \subseteq \mathbb{R}^n$ is convex + closed then so is $-A := \{-x : x \in A\}$ → easy to verify

Also, if A, B convex + closed then $A+B := \{a+b : a \in A, b \in B\}$ is convex + closed → Also easy + on hw

Follows that $A-B$ is convex + closed

By $A \cap B = \emptyset$, know $0 \notin A-B$

$\{0\}$ is convex (+ compact) so by convex separation $\exists v$ s.t.

$$\langle 0, v\rangle < \langle a-b, v\rangle, \quad \forall a \in A, b \in B$$

$$\Updownarrow$$

$$\langle b, v\rangle < \langle a, v\rangle \quad \forall a \in A, b \in B \rightarrow A, B \text{ strictly separated}$$

Given $x=(\dots,x_{i-1},x_i,x_{i+1},\dots)$, $y=(\dots,x_{i-1},x_{i+1},\dots)$ is the result of projecting out i

Let $\mathrm{Proj}(x,I)$ be the result of projecting out all $i\in I$

Claim: Given any $I\subseteq[n]$, $A\in\mathbb{R}^{m\times n}$, $b\in\mathbb{R}^m$, $\{\mathrm{Proj}(x,I): Ax\le b\}$ is a polyhedra → "Projection of a polyhedra is a polyhedra"

just state

Suffices to show for $|I|=1$ by induction; then follows by F-M elimination to eliminate x_i

Claim: Given linear $A:\mathbb{R}^n\to\mathbb{R}^m$, $I:=\{Ax: x\in\mathbb{R}^n\}$ is a polyhedra → "Image of linear fn. is a polyhedra"

Consider $I':=\{(x,Ax): x\in\mathbb{R}^n\}\subseteq\mathbb{R}^{n+m}$; let $A'=(A\ \ {-I})$ so $I'=\{y: A'y=0\}$ is a polyhedra

But $I:=\{x:(x,Ax)\in I'\}$ is just the projection of I, so follows by previous statement

Proof of Farkas w/ Polyhedral Separation

Already showed if $\exists\,\lambda\in\mathbb{R}^m_{\ge 0}$ s.t. $A^T\lambda=0$ and $\langle\lambda,b\rangle<0$ then $\exists x$ s.t. $Ax\le b$

WTS if $\nexists x$ s.t. $Ax\le b$ then $\exists\,\lambda\in\mathbb{R}^m_{\ge 0}$ s.t. $A^T\lambda=0$ and $\langle\lambda,b\rangle<0$

Suppose $\nexists x$ s.t. $Ax\le b$

Let $I:=\{Ax: x\in\mathbb{R}^n\}$ be the image of A, $B:=\{b': b'\le b\}$

I is a polyhedra b/c it is the image of a linear fn., B is a polyhedra b/c $B=\{b': Ib'\le b\}$

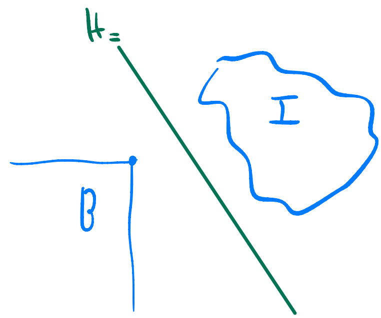
By assumption $B\cap I=\emptyset$

Also $0\in I$ (b/c $A0=0$) and $b\in B$ so I,B non-empty so by polyhedral separation

$\exists\,\lambda\in\mathbb{R}^n, a\in\mathbb{R}$ s.t.

$\langle b',\lambda\rangle<a \quad \forall\, b'\in B$

$\langle Ax,\lambda\rangle\ge a \quad \forall\, x\in\mathbb{R}^n$



But want $a=0$ →

$\langle b,\lambda\rangle<0 \quad \forall\, b'\in B$

$0\in I$ so $0=\langle 0,\lambda\rangle=\langle A0,\lambda\rangle\ge a$ so $a\le 0$

$A^T\lambda=0$

$\langle Ax,\lambda\rangle\ge 0\ \forall x\in\mathbb{R}^n$ b/c $\forall\, c\ge 0$ have $\langle Acx,\lambda\rangle\ge a$ so $\langle Ax,\lambda\rangle\ge\frac{a}{c}$ so $\langle Ax,\lambda\rangle\ge 0\ \forall x$

But $\langle Ax,\lambda\rangle=\langle A^T\lambda,x\rangle$ so $\langle A^T\lambda,x\rangle\ge 0\ \forall x$ → Only possible if $A^T\lambda=0$

Lastly $\lambda\ge 0$

AFSOC $\exists\, i$ s.t. $\lambda_i<0$

Then $\langle\lambda,b'\rangle>0$ for any b' w/ $b'_i\to-\infty$, contradicting $\langle b',\lambda\rangle<0$

Proof of Strong Duality w/ Farkas

Already saw $P \leq D$ so

AFSOC that $P < D$

So $\nexists x$ s.t. $Ax \leq b$ and $\langle c, x \rangle \geq D$

$\Updownarrow$

$\nexists x$ s.t. $\underbrace{\begin{pmatrix} A \\ -c^T \end{pmatrix}}_{\hat{A}} x \leq \underbrace{\begin{pmatrix} b \\ -D \end{pmatrix}}_{\hat{b}}$

By Farkas $\exists$ $y \in \mathbb{R}^{m+1}_{\geq 0}$ s.t. $\hat{A}^T y = 0$ and $\langle \hat{b}, y \rangle < 0$

Write y as $(\underset{\in \mathbb{R}^m}{\lambda}, \underset{\in \mathbb{R}}{z})$

So $0 = \hat{A}^T y = A^T \lambda - z \cdot c^T$ and $\langle \lambda, b \rangle - D \cdot z < 0$

Have $z > 0$

AFSOC $z = 0$

Then $0 = A^T \lambda - z \cdot c^T = A^T \lambda$ so $A^T \lambda = 0$

Also $\langle \lambda, b \rangle = \langle \lambda, b \rangle - D \cdot z < 0$

So Farkas gives $\nexists$ x s.t. $Ax \leq b$, Contradicting Primal Feasible

Consider $\frac{1}{z} \cdot \lambda \geq 0$

$A^T \lambda - z \cdot c^T = 0$ so $A^T \frac{\lambda}{z} = c^T$ so $\frac{1}{z} \cdot \lambda$ dual feasible

But $\langle b, \frac{1}{z} \cdot \lambda \rangle = \frac{1}{z} \cdot \langle b, \lambda \rangle < \frac{1}{z} \cdot D \cdot z = D$, Contradicting D defn.