Today

1) Intro to Convex Geometry
2) Equivalence of Corners
3) Polytopes as Convex Hulls

Recall

A set of the form $K = \{x : Ax \leq b\}$ is called a <u>Polyhedron</u>

If $\underbrace{\exists \Delta \in \mathbb{R} \text{ s.t. } d(x,y) \leq \Delta \ \forall x,y \in K}_{\hookrightarrow K \text{ is "bounded"}}$, then K is called a <u>Polytope</u>

Lemma: Given $x \in K$, $w \in Ker(A_T)$ $\exists \epsilon > 0$ s.t. $\underbrace{x \pm \epsilon \cdot w}_{\{x+\epsilon w,\, x-\epsilon w\}} \subseteq K$

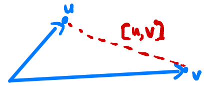
~~and if $\langle w, a_i \rangle \neq 0$ for some $a_i \in rows(A)$ then $Tight_A(x) \subsetneq Tight_A(y)$~~
~~for some $y \in$~~ $\cancel{\underbrace{x \pm \epsilon \cdot w}_{\hookrightarrow \{x+\epsilon w,\, x-\epsilon w\}}}$ not needed

x is a <u>basic feasible solution (BFS)</u> of K iff $x \in K$ and $rank(Tight_A(x)) = n$

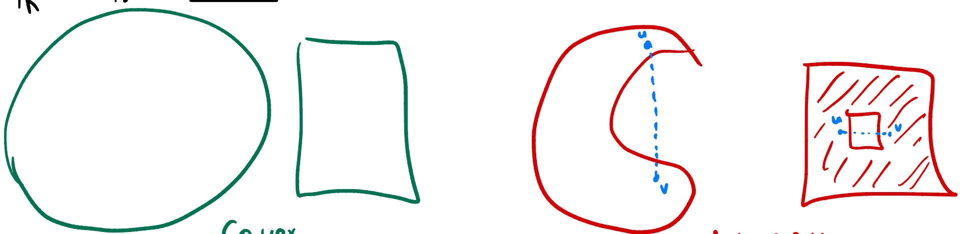
Intro to Convex Geometry

Given $u,v \in \mathbb{R}^n$, $p \in [0,1]$, $w = p\cdot u + (1-p)\cdot v$ is called a **convex combination** of u,v

$$[u,v] := \{p\cdot u + (1-p)\cdot v \,:\, p \in [0,1]\}$$



$K \subseteq \mathbb{R}^n$ is **convex** if $u,v \in K \to [u,v] \subseteq K$



Convex

Not convex

Fact: Polyhedron $K = \{Ax \le b\}$ is convex

Suppose $u,v \in K$ and $p \in [0,1]$

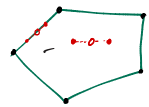
$$A(p\cdot u + (1-p)\cdot v) = p\,A(u) + (1-p)\cdot A(v) \le p\cdot b + (1-p)\cdot b \le b$$

So $p\cdot u + (1-p)v \in K$ so $[u,v] \subseteq K$ so K convex

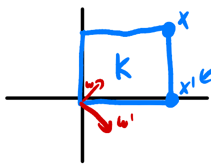
Equivalent Definitions of "Corners"

x is a <u>basic feasible solution (BFS)</u> of K iff $x \in K$ and $\text{rank}(\text{Tight}_A(x)) = n$

x is an <u>extreme point</u> of K iff $x \in K$ and $x \in [y,z]$ for $y,z \in K \to y = z$



x is a <u>vertex</u> of K iff $x \in K$ and $\exists\ w$ s.t. $\langle w,x\rangle > \langle w,y\rangle\ \forall y \in K, y \neq x$



not always true that $x = w$

Fact: $x \in K = \{Ax \le b\}$ is a BFS iff it is a vertex iff it is an extreme point

BFS $\to$ vertex

Let $x \in K$ be a BFS and fix basis B of $\text{Tight}_A(x)$; x is unique point in K tight for B — b/c $A_B x = b_B$ and A_B full rank (Problem 7 LA review)

Let $w := \sum_{a_i \in B} a_i \to$ Since $B \subseteq \text{Tight}_A(x)$, have $\langle x,w\rangle = \sum_{i: a_i \in B} b_i$

OTOH for any $y \in K$ s.t. $y \neq x$, have $\langle y,w\rangle < \sum_{i: a_i \in B} b_i$ (Since y not tight for all constraints $\langle a_i, y\rangle \le b_i$, $a_i \in B$

extreme point $\to$ BFS

By contrapositive $\to$ Suppose x not a BFS so $\text{rank}(\text{Tight}(x)) < n$ so $\exists\, w \neq 0 \in \text{Ker}(A_{\text{Tight}(x)})$

But then by lemma have $\exists\, \epsilon > 0$ s.t. $x \pm \epsilon \cdot w \subseteq K$ so $x = \frac{1}{2}(x + \epsilon \cdot w) + \frac{1}{2}(x - \epsilon \cdot w)$ — $\neq 0$

so x not an extreme point

vertex $\to$ extreme point

By contrapositive $\to$ Suppose x not an extreme point so $x = p \cdot u + (1-p)v$ for some $p \in (0,1)$, $u \neq v$, $u, v \in K$

For any $w \in \mathbb{R}^n$ have $\langle w, x\rangle = p\langle w,u\rangle + (1-p)\langle w,v\rangle$

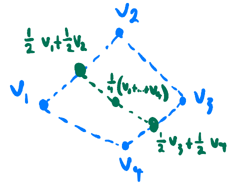
$\hookrightarrow$ Follows that either $\langle w,u\rangle \geq \langle w,x\rangle$ or $\langle w,v\rangle \geq \langle w,x\rangle$ so x not a vertex

Polytopes as Convex Hulls

Given $V=\{v_1,v_2,\dots\}\subseteq \mathbb{R}^n$, p s.t. $p_i\in[0,1]$, $\sum_i p_i=1$, $w=\sum_i p_i v_i$ is a **convex combination** of V

Given $V=\{v_1,v_2,\dots\}\subseteq \mathbb{R}^n$, the **convex hull** of V is

$$\mathrm{Con}(V) := \{u : u \text{ is a convex combination of } V\}$$



Surprisingly hard to show

Fact: If K is a polytope w/ BFSs/vertices/extreme points V then $K=\mathrm{Conv}(V)$