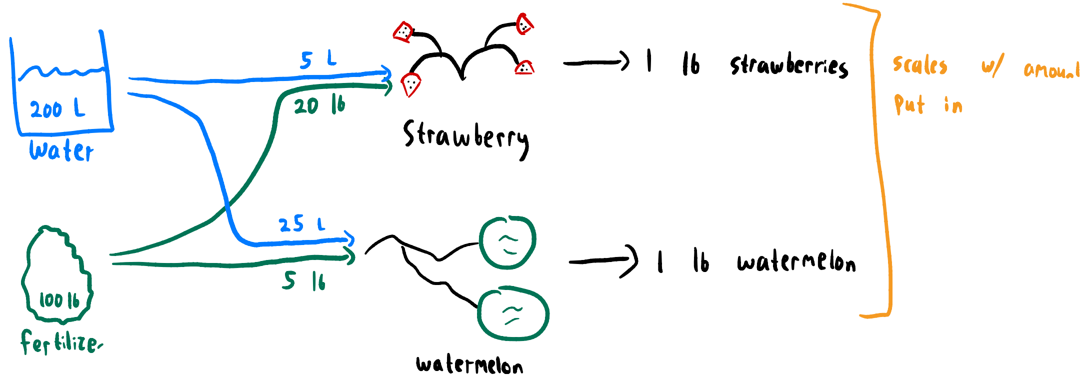
Today

1) LP Feasibility
2) LP Search via Feasibility ← break
3) LP Optimization via Search

Admin

- Names
- Hw2 in, hw3 out
- Exam next week
  - closed book
  - Hw problems



Can I grow 10 lb of strawberries and 5 lb of watermelon?

$x_1 :=$ lbs of strawberries grown

$x_2 :=$ lbs of watermelon grown

$x := (x_1, x_2)$

$\exists x_1, x_2$ s.t.

$$\begin{aligned} -x_1 &\le -10 \\ -x_2 &\le -5 \\ 5x_1 + 25x_2 &\le 200 \\ 20x_1 + 5x_2 &\le 100 \end{aligned}$$

$a_3 = (5, 25)$ $b = (-10, -5, 200, 100)$

$\exists x$ s.t.

$$\underbrace{\begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 5 & 25 \\ 20 & 5 \end{pmatrix}}_{A} x \overset{\text{coordinate wise}}{\le} \underbrace{\begin{pmatrix} -10 \\ -5 \\ 200 \\ 100 \end{pmatrix}}_{b}$$

$K \ne \emptyset$? where $K := \{x : Ax \le b\}$

Linear Programming LP Feasibility Problem:

Inequalities: Given $\underbrace{\left.\begin{matrix} a_{11}, a_{12}, \dots, a_{1n}, b_1 \\ a_{21}, a_{22}, \dots, a_{2n}, b_2 \\ \dots \\ a_{m1}, a_{m2}, \dots a_{mn}, b_m \end{matrix}\right]}_{\text{an "LP"}} \in \mathbb{R}$ decide if $\exists x \in \mathbb{R}^n$ s.t. $a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \le b_i \ \forall i$

Given $a_1, \dots, a_m \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ decide if $\exists x \in \mathbb{R}^n$ s.t. $\langle x, a_i \rangle \le b_i \ \forall i$

$\Updownarrow$

Linear Algebraic: Given $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, decide if $\exists x \in \mathbb{R}^n$ s.t. $Ax \le b$

$\Updownarrow$

Geometric: Given $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $K = \{x : Ax \le b\}$, decide if $K \ne \emptyset$

$x \in \mathbb{R}^n$ is <u>feasible</u> if it satisfies all inequalities; LP is <u>feasible</u> if $\exists$ feasible x

Theorem: Can solve LP feasibility in time $O(m^{2^n})$

Idea: Modify LP in feasibility-preserving way until obvious if feasible

↳ E.g. initial gardening Problem feasibility unclear

but Clearly $5 \le -5$ not feasible

$\forall x \in \mathbb{R}^n$

Can be a fn. of x

① $\forall c \in \mathbb{R} \; [\langle a_i, x \rangle \le b_i \text{ iff } \langle a_i, x \rangle + c \le b_i + c]$

Only if

② $\forall \lambda \ge 0 \; [\langle a_i, x \rangle \le b_i \; \cancel{\text{iff}} \; \lambda \langle a_i, x \rangle \le \lambda \cdot b_i]$

③ $\langle a_i, x \rangle \le b_i \text{ iff } -\langle a_i, x \rangle \ge -b_i$

Say x_i is <u>isolated</u> in an inequality if it is of form $\bullet \le x_i$ or $x_i \le \bullet$

↳ Can apply ①, ②, ③ to isolate x_i while preserving feasibility

Fourier-Motzkin Elimination

For $i = 1, 2, \dots$

Isolate x_i in all inequalities containing it

For each pair of inequalities of form $\ell \le x_i$, $x_i \le r$

Add $\ell \le r$ to inequalities

Possibly fns of x w/o x_i

Delete all inequalities containing x_i

Return feasible iff final inequalities feasible

	Isolate x_1	Replace x_1	Isolate x_2	Replace x_2
$-x_1 \le -10$	$10 \le x_1$	$10 \le 40 - 5x_2$	$x_2 \le 6$	$5 \le 6$
$-x_2 \le -5$	$-x_2 \le -5$	$10 \le 5 - x_2$	$x_2 \le -5$	$5 \le -5$
$5x_1 + 25x_2 \le 200$	$x_1 \le 40 - 5x_2$	$-x_2 \le -5$	$5 \le x_2$	Not feasible!
$20x_1 + 5x_2 \le 100$	$x_1 \le 5 - 5x_2$			

Correctness of FM

Suffices to show isolating, eliminating leaves feasibility unchanged

Isolating an x_i: only applying (1), (2) or (3) so feasibility preserved

Eliminating an x_i:

Suppose $\exists x$ satisfying all inequalities before eliminating x_i

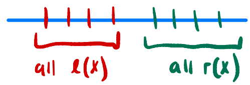
Then $\ell(x) \le x_i$ and $x_i \le \ell(x)$ $\forall$ inequalities of this form

So $\ell(x) \le r(x)$ so x satisfies all new inequalities

Suppose $\exists x$ satisfying all inequalities after eliminating x_i

Then $\ell(x) \le r(x)$ $\forall \ell, r$

So $\exists c \in [\ell(x), r(x)]$ $\forall \ell, r$



So if $x_i = c$, have $\ell \le x_i$, $x_i \le r$ $\forall \ell, r$; rest of x unchanged

Runtime of FM

Iteration	0	1	2	...	i	...	n
# Inequalities	m	$\le m^2$	$\le m^4$	...	m^{2^i}	...	m^{2^n}

So runtime is $O\left(m^{2^n}\right)$

LP Search Problem: Given A, b output x s.t. $Ax \leq b$ or report no such x

Going forward let $\underline{N} := m + n$

Theorem: If can solve LP feasibility in $T(N)$ time
then can solve LP search in $\text{poly}(N) \cdot T(N)$ time

Can model the following constraints w/ LPs

$\forall x \ \langle a_i, x\rangle + b_i \leq \langle a_i', x\rangle + b_i'$ iff $\langle a_i - a_i', x\rangle \leq b_i' - b_i$

$\forall x \ \langle a_i, x\rangle \geq b_i$ iff $\langle -a_i, x\rangle \leq -b_i$

$\forall x \ \langle a_i, x\rangle = b_i$ iff $\langle a_i, x\rangle \leq b_i$ and $\langle a_i, x\rangle \geq b_i$

An LP is in equational form if its constraints are $Ax = b$, $x \geq 0$

Claim: If can solve equational form LP search in $T(N)$ time
then can solve LP search in $T(O(N))$ time

$x^+ =$ ≥ 0 coords of x

$x^- =$ negative of <0 coords of x

$$\exists x \in \mathbb{R}^n \text{ s.t. } Ax \leq b$$

$x = x^+ - x^-$

iff

$$\exists x^+, x^- \in \mathbb{R}^n \text{ s.t. } Ax^+ - Ax^- \leq b \ \text{ and } \ x^+, x^- \geq 0$$

$s = b - A(x^+ - x^-)$

trivial

iff

$$\exists x^+, x^- \in \mathbb{R}^n, s \in \mathbb{R}^m \text{ s.t. } Ax^+ - Ax^- + \underbrace{I}_{m \times n \text{ identity}} \cdot s = b \ \text{ and } \ x^+, x^-, s \geq 0$$

↳ An equational form LP $\underbrace{\begin{pmatrix} A & -A & I \end{pmatrix}}_{A' \in \mathbb{R}^{m \times 2n+m}} \begin{pmatrix} x^+ \\ x^- \\ s \end{pmatrix} = b$, $x^+, x^-, s \geq 0$

To solve search on $Ax \leq b$, solve above equational search, let $x = x^+ - x^-$

m constraints, $2n+m$ variables so $T(2n+2m) = T(O(n))$ time

Can almost solve equational form LP search w/ GE

Fact: Given $m \times n$ matrix A and $b \in \mathbb{R}^m$, let $K' = \{x : Ax = b\}$

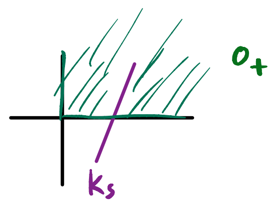
Can output $x \in K'$ or report K' empty in $\mathrm{Poly}(N)$ (Gaussian Elimination)

Let $\underline{O_+} := \{x : x \geq 0\}$

Given $S \subseteq [n]$, let $\underline{K_S} = \left\{ x : \begin{matrix} Ax = b \\ x_i = 0 \;\; \forall i \in S \end{matrix} \right\}$ → Can find $x \in K_S(A)$ w/ GE

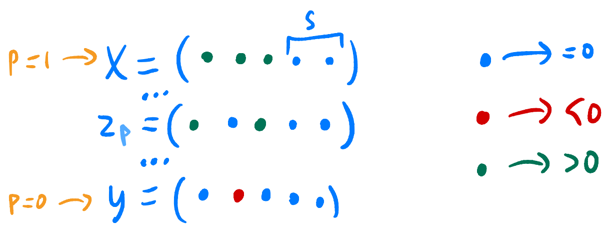
Lemma: If $K_S \cap O_+ \neq \emptyset$ and $K_S \setminus O_+ \neq \emptyset$

then $\exists \; i \notin S$ s.t. $K_{S+i} \cap O_+ \neq \emptyset$



Let $x \in K_S \cap O_+$ and $y \in K_S \setminus O_+$

For $p \in [0,1]$, let $z_p := p \cdot x + (1-p) \cdot y$



$\forall p$ have $A(z_p) = b$ b/c $A(z_p) = A(p \cdot x + (1-p) \cdot y) = p \cdot Ax + (1-p) Ay \leq b$

But $\exists \; p$ s.t. $z_p \geq 0$ and $(z_p)_i = 0$ for some $i \notin S$

So $\exists \; p, i \notin S$ s.t. $z_p \in K_{S+i} \cap O_+$ so $K_{S+i} \cap O_+ \neq \emptyset$

Lemma: If can solve (equational form) LP feasibility (in time $T(N)$)

then can solve equational form LP search (in time $Poly(N) \cdot T(O(N))$)

To find $x \in K := \{x : Ax = b, x \geq 0\}$

Equational Search via Equational Feasibility

$S \leftarrow \emptyset$

Repeat $n+1$ times:

$\{x : Ax = b, x_i = 0 \ \forall i \in S\}$

Compute $y \in K_S$ (via GE)

If $y \in O_+$, return y

If $\exists i \notin S$ s.t. $K_{S+i} \cap O_+ \neq \emptyset$ (via feasibility)

$S \leftarrow S + i$

Else Report $K = \emptyset$

Correctness

If y returned then $y \in K_S \cap O_+ \subseteq K$ so anything returned is feasible

So WTS if $K \neq \emptyset$ then return something

If $K \neq \emptyset$ then initially $K_S \cap O_+ \neq \emptyset$

So by induction + lemma, in each iteration $S \leftarrow S+i$

So either return a y or iterate $n+1$ times and return 0

Runtime

Each feasibility instance has $\leq m+n$ constraints

↳ Solved n^2 times for $n^2(T(O(N)))$

Each GE instance has n variables, $\leq m+n$ constraints

↳ Solved n times for $Poly(N)$ time

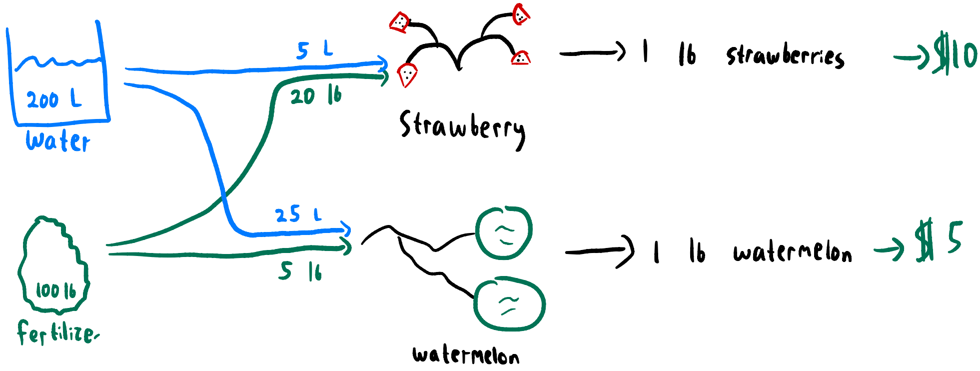
Search → Feasibility theorem then follows

(Equational) LP Feasibility → Equational LP search → LP Search

$T(N)$ → $Poly(N) \cdot T(O(N))$ → $Poly(N) \cdot T(O(N))$

LP Optimization Problem: Given $A, b, c \in \mathbb{R}^n$ output x maxing $\langle c, x \rangle$ s.t. $Ax \le b$ or report no such x

get min by negating c



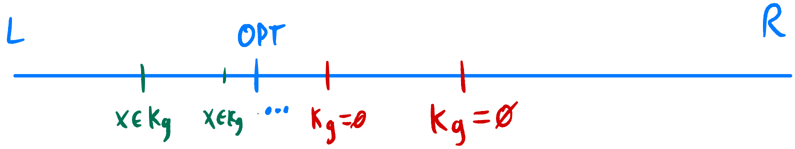
$$\max\ 10x_1 + 5x_2 \ \text{s.t.}$$
$$5x_1 + 25x_2 \le 200$$
$$20x_1 + 5x_2 \le 100$$

Theorem: If can solve LP Search in $T(N)$ time then can solve LP optimization in $\mathrm{Poly}(N) \cdot T(N)$ time

(Assuming 1) Have L, R s.t. $L \le OPT \le R$, $|L - R| \le \exp(\mathrm{Poly}(N))$
2) Optimal value $\in \mathbb{Z}$)

Let $K_g := \left\{ x : \begin{array}{l} Ax \le b \\ \langle c, x \rangle \ge g \end{array} \right\}$ for $g \in \mathbb{Z}$

Binary search $g \in [L, R] \cap \mathbb{Z}$



Getting Rid of Assumptions

$OPT \notin \mathbb{Z}$?

→ If optimal value ∞ — Can poly-time compute $M \in \mathbb{R}$ s.t. if $\exists x \in K$ w/ $\langle x, c \rangle \geq M$ then $OPT = \infty$ (LP is "unbounded")

E.g. max x_1 s.t. $x_1 \geq 0$

→ O/w After poly(n, m) BS rounds, can round g to OPT w/ # theory in poly-time

L, R?

→ Poly-time computable

Summarizing

LP Feasibility		LP Search		LP Optimization
Decide if $\exists x$ s.t. $Ax \leq b$	→	Find x s.t. $Ax \leq b$	→	Find x maxing $\langle c, x \rangle$ s.t. $Ax \leq b$
$\underbrace{T(N)}_{\leq N^{2^N}}$		$\text{Poly}(N) \cdot T(O(N))$		$\text{Poly}(N) \cdot T(O(N))$