Lecture 3: LLL, Homework

1. **For some practice with mutual dependence**

   Our definition of mutual independence has
   Definition 1 (Mutual Independence)**.** *Event A is independent of events $\beta = \{B_1, \ldots, B_m\}$ if*

   $$\Pr[A|\beta'] = \Pr[A]$$

   for all $\beta' \subseteq \beta$

   Show that this is equivalent to the following definition of mutual independence (noting the difference in β in this new definition).
   Definition 2 (Mutual Independence')**.** *Event A is independent of events $\beta = \{B_1, \ldots, B_m, \bar{B_1}, \ldots, \bar{B_m}\}$ if*

   $$\Pr[A|\beta'] = \Pr[A]$$

   for all $\beta' \subseteq \beta$

2. **For some practice with applying the LLL**

   Suppose $11n$ points are placed around a circle for $n \in \mathbb{N}$. Call this set S.
   Definition 3 (Valid Coloring)**.** *A coloring (i.e. an assignment of colors to points) of $11n$ points is valid if it uses n colors, each one exactly 11 times.*

   Call a subset of our $11n$ points **rainbow** if each point is given a different color.

   Say that two points $x, y \in S$ are **adjacent** if either there are no points in S between x and y or there are no points between y and x.

   Prove that:
   Lemma 1. *In **any** valid coloring there is a rainbow set $S' \subseteq S$ of n points such that no two points in S' are adjacent.*

3. **For some practice with applying the LLL**

   We will apply the LLL to prove that certain edge-colorings of certain graphs always exist. Define the following notions of proper and acyclic edge-colorings.
   Definition 4. *An edge-coloring of G is an assignment of edges to colors.*
   Definition 5 (Proper Edge-Coloring)**.** *An edge-coloring is proper if no vertex is incident to two edges of the same color.*
   Definition 6 (Acyclic Edge-Coloring)**.** *An edge-coloring is acyclic if every two colors induce a forest.*

   Define $a(G)$ as the minimum number of colors in an acyclic proper edge-coloring of graph G.

   Alon, Sudakov and Zaks conjectured that $a(G) \leq \Delta(G) + 2$ where Δ is the max degree of G. They managed to show the following weaker claim which assumes the girth of a graph is large (defined below1).
   Definition 7 (Girth)**.** *Define the girth of graph G, $g(G)$, as the length of the shortest cycle in G.*

   Using the LLL, show that
   Theorem 1. *Provided $g(G) \geq \Omega(\Delta \log \Delta)$ we have $a(G) \leq \Delta(G) + 2$.*

Hint: recall that Vizing's theorem states that every graph G has a proper edge-coloring $C : E \to [\Delta]$ using $\leq \Delta + 1$ colors. Consider starting with Vizing's coloring and then changing the color of each edge with probability p to a new $(\Delta + 2)$th color, say red. Define bad events: A_B as two adjacent edges are colored red; A_C as a bichromatic cycle in C has no edges colored red; A_D as the cycle D is bichromatic in red and one of the colors of C.

4. **To see how the symmetric and asymmetric LLL relate**

   There is an "asymmetric" version of the LLL which we didn't have time to get to today. This lemma states
   Lemma 2 (Asymmetric LLL)**.** *Given bad events $A_1, \ldots, A_m$ and a dependency graph as before if there exists an assignment to reals $x_i \in [0, 1)$ such that $p_i < x_i \cdot \prod_{A_j \in \Gamma(A_i)} (1 - x_j)$ for every A_i then $P[\wedge_i \bar{A}_i] > 0$.*

   Show that the the asymmetric LLL implies the symmetric LLL for the case where all p_is are equal?

5. **A question I don't know the answer to**

   Recall in class that we saw the following two lemmas:
   Lemma 3. *A k-SAT instance where each variable occurs in strictly fewer than $\frac{2^k}{ek}$ clauses is satisfiable.*
   Lemma 4. *For every k there exists a k-SAT formula where every variable occurs in 2^k clauses which is not satisfiable.*

   A nice question which Ziye asked is whether there exist k-SAT formulas where every variable occurs in at most $\frac{2^k}{k}$ clauses which are not satisfiable. Give some thought to this question as well as, more broadly, which of 2^k and $\frac{2^k}{ek}$ are tight for the above bounds.